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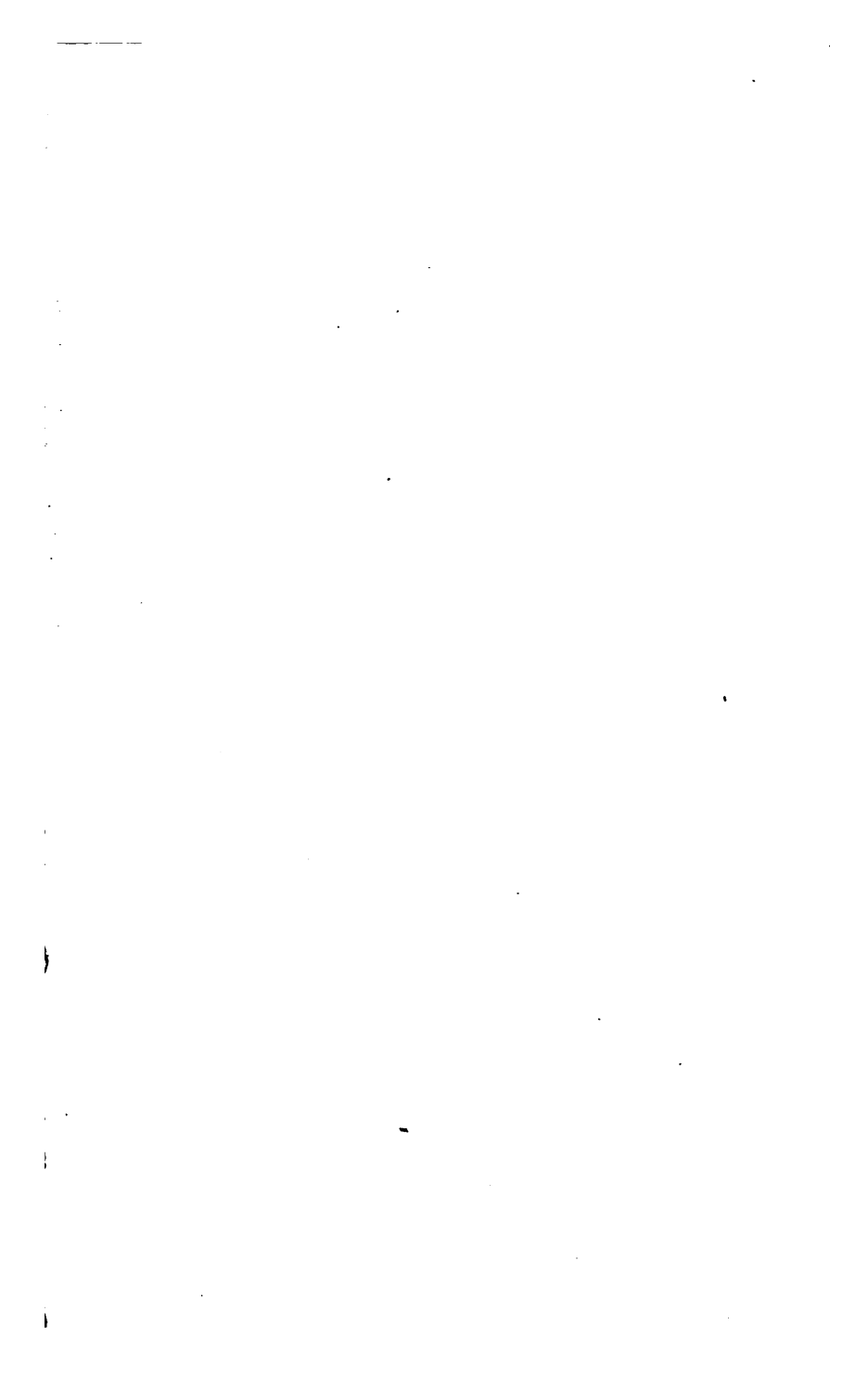


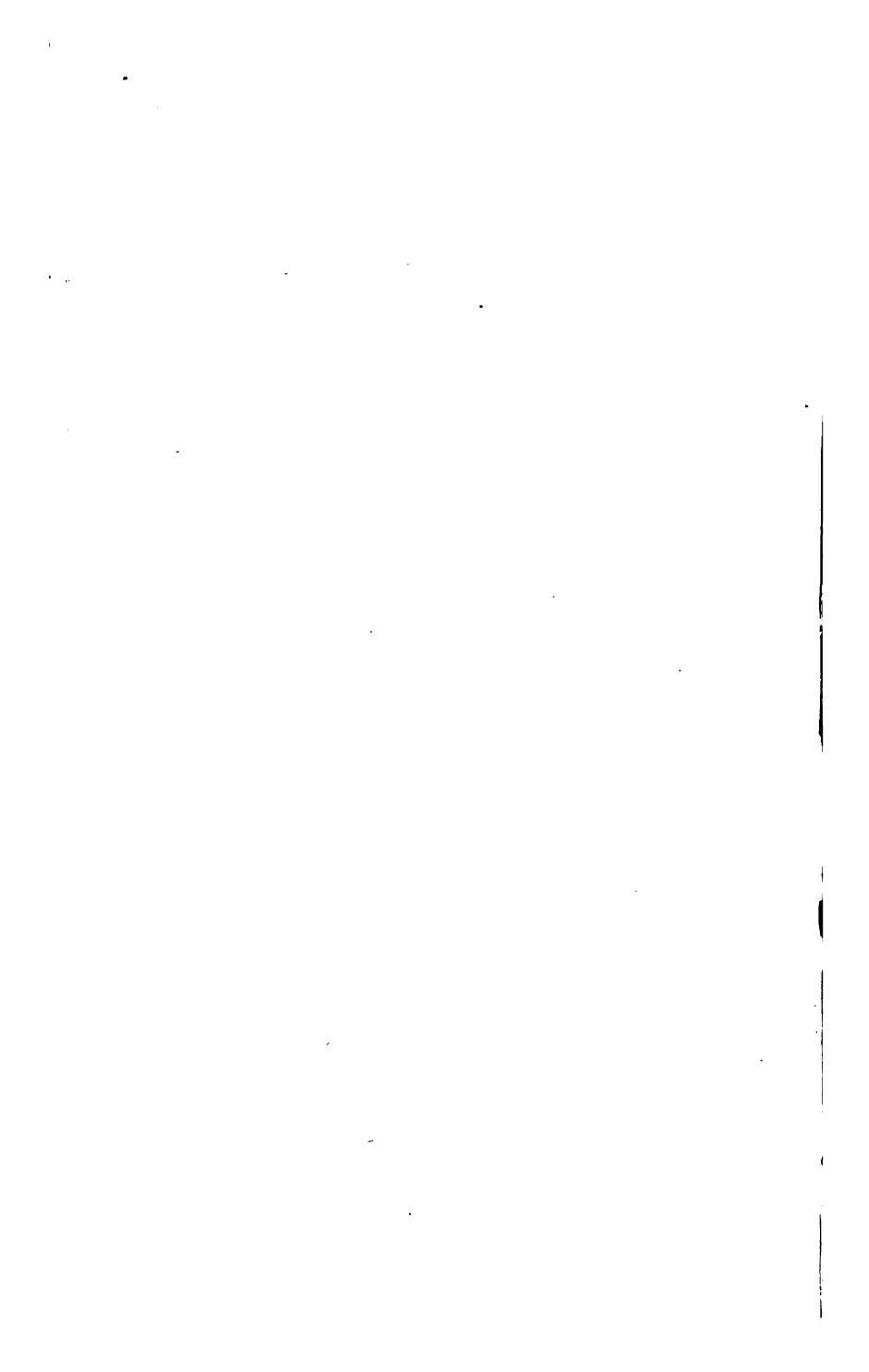
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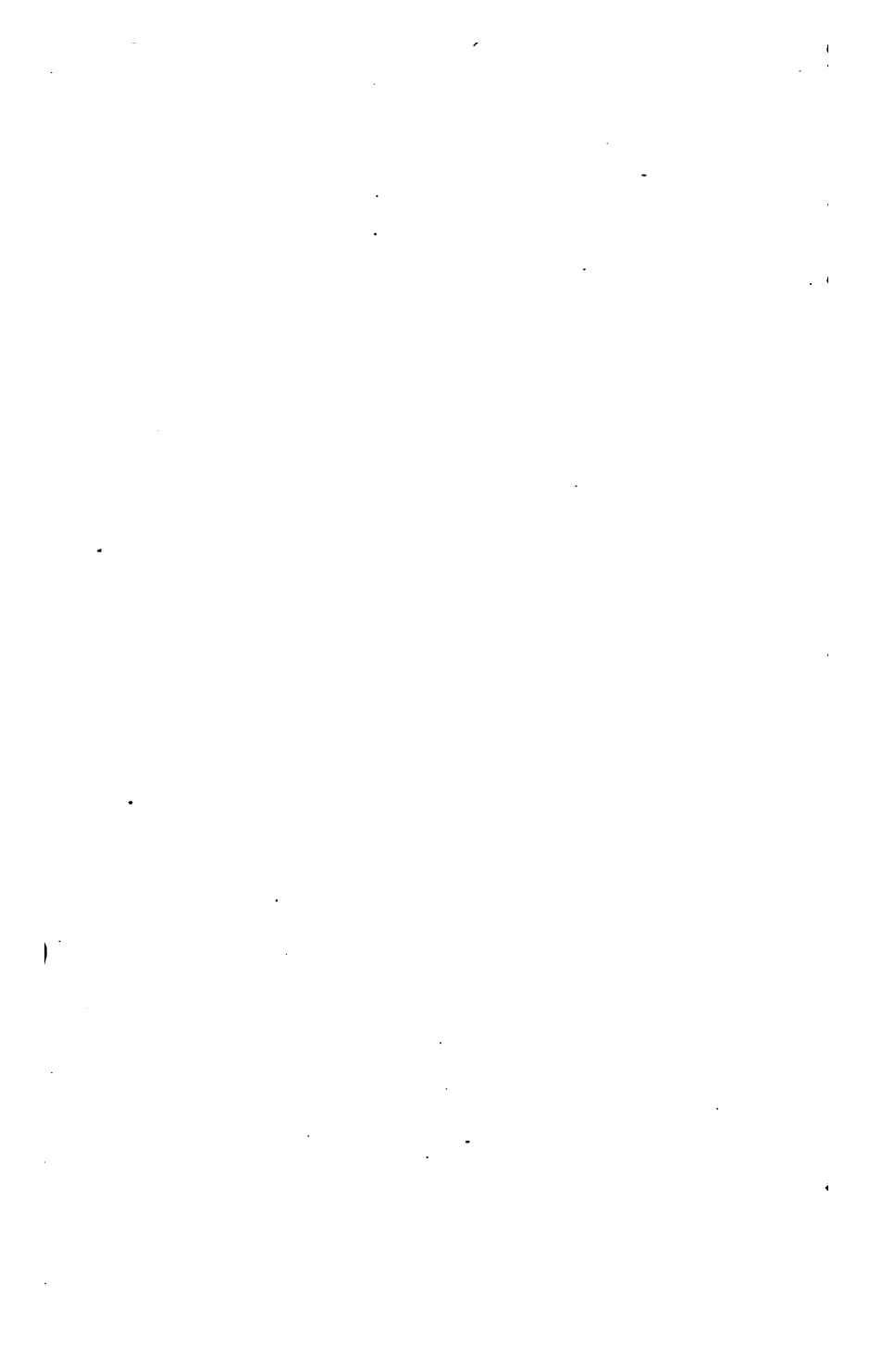


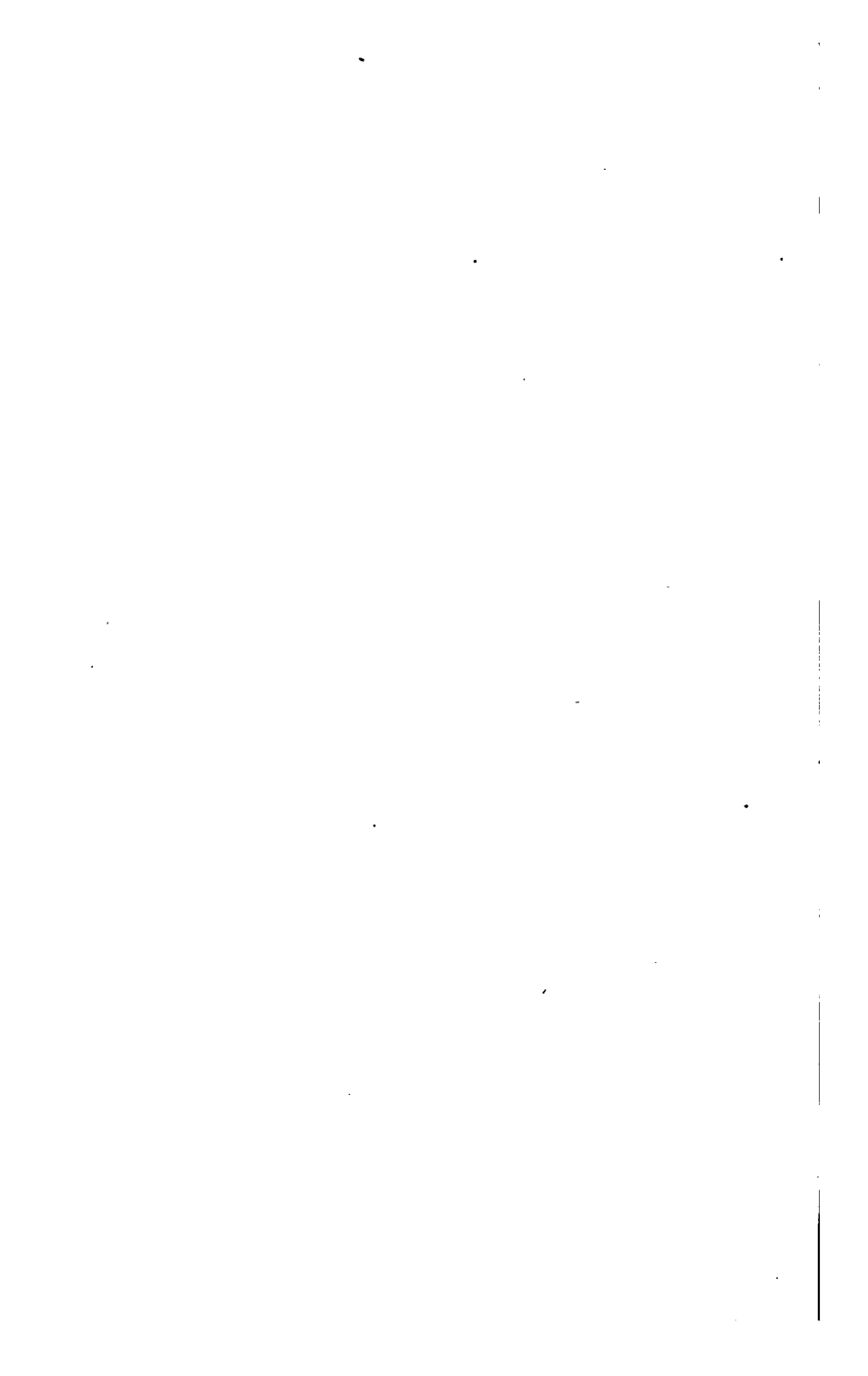
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ELEMENTARY

A L G E B R A.

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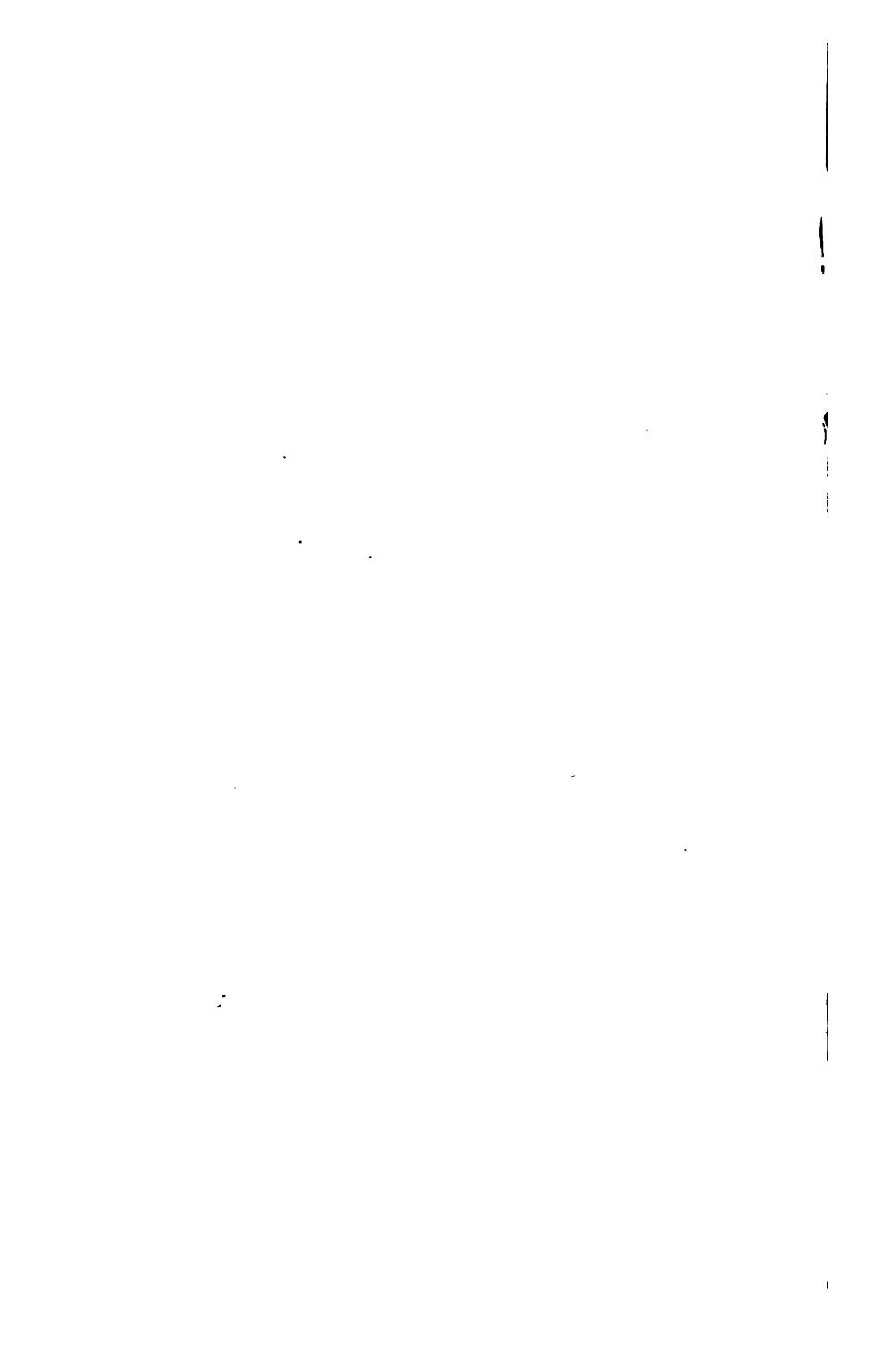
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PREFACE.

To render the Science of Algebra intelligible to pupils whose minds are yet unaccustomed to such studies, is not an easy task. For should the instructor subject every principle, as it is announced, to a rigorous demonstration, he will very probably not be comprehended; while, on the other hand, inconclusive reasoning is worse than none at all.

To obviate this difficulty has been the main object of the present treatise. The beginner will here be furnished with such proofs as are suited to his capacity; examples will afford new light to what might be otherwise obscure; with regard to the operations founded on higher principles he will, for the present, content himself with merely practical rules, exemplified in the same manner.

With a mind thus gradually led on to strict mathematical discussion, he may then resume his course with profit, by the aid of a treatise now in preparation, which is intended as a sequel to this, and by more exact and thorough investigation, complete his study of Algebra.



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ALGEBRA.

PRELIMINARY ARTICLE.

§ 1. A NUMBER, for example 7 or 8, is used to signify or represent seven or eight different beings; for instance, seven or eight lamps, seven or eight boats. But the figure 7 cannot express more than seven, and the figure 8 no more than eight.

Different signification of numbers and algebraic symbols.

Algebraic *symbols* have a general signification, and the same symbol may be used to represent five, or seven, or a hundred different objects.

The letters a , b , &c. of the Latin, and α , β , &c. of the Greek alphabet, are employed as algebraic symbols; and therefore a , for instance, may be used equally well to represent seven stones, or nine rods, or fifty rings, &c. The same symbol is frequently used to represent different quantities of the same order; and then, to distinguish the different significations of the symbol, we add to it one or more accents, as follows:

a' , a'' , a''' , ... a^n ,

and read—*a prime, a second, &c.*

or else, instead of the accents, numbers may be used, as follows:

a , a_1 , a_2 , ... a_n ,

and read—*a one, a two, &c.*

§ 2. Algebraical operations are the same as with numbers, but on account of the more general signification of algebraic symbols, algebraic computation is also more general than the arithmetical.

Algebraic operations the same as arithmetical ones, but more general.

Simple and compound symbols.

§ 3. Algebraic symbols are either simple or compound. A simple symbol is, for example, a or b ; but if a or b are connected with some other symbol, or with numbers, this union causes them to become compound symbols. Symbols can be connected together in many ways, and so likewise symbols and numbers. A symbol, for example, can be added to itself several times, or multiplied by itself several times.

Coefficient and exponent.

§ 4. When a symbol, for example b , is added to itself two or three times, instead of taking it as many times as it is to be added, we write the symbol only once, placing before it the number which indicates the number of times that symbol is taken; so in our case we should have $2b$ or $3b$. The number placed before the symbol is called *coefficient*. The coefficient may be a fraction as well as a whole number, and also another algebraic symbol. When a symbol is multiplied by itself three or four, or more times, we express it likewise by a number placed, not before the symbol, like the coefficient, but above it. The expression, for example, c^4 , signifies that c is multiplied by itself four times; and if the same *exponential* expression would be multiplied again by c or by c^2 , it would become equivalent to c^5 or c^6 , &c. The number 4 is called *exponent*. The exponent, like the coefficient, can be a whole number, or a fraction, or an algebraic symbol.

Signs, plus and minus.

§ 5. When a symbol is to be added to another, or when several symbols are to be added together, we write them in succession, placing between them the sign $+$, termed *plus*. So, for instance, if a , b , c are to be added together, we write $a+b+c$, and read a plus b plus c .

When a symbol is to be taken or subtracted from another, for example, the number 7, represented by b , from the number 15, represented by a , we write first the symbol a and then b , placing between them the sign $-$, termed *minus*. So if n

is to be taken from m , we will write $m - n$, and read m minus n .

§ 6. An isolated symbol, either simple or compound, is called *monomial*, or *term*. Monomial and polynomial. Two symbols linked together by the sign plus, or minus, constitute a *binomial*, as, for instance, $a + b$, or $m - n$. Three symbols connected in equal manner form a *trinomial*, as $a + b + c$; and generally any number of terms thus joined is called *polynomial*.

§ 7. A term a can be multiplied by another term b , in equal manner as a number may be multiplied by another. Signs of multiplication and division. In such case we write either ab , and read simply a , b , or

$$a \times b, a . b$$

and read a multiplied by b .

The sign $\times$ and $.$ are termed signs of multiplication.

If the term a is to be divided by the term b , then we write $a : b$ or $\frac{a}{b}$ and read a divided by b . The double dot and line are the signs of division.

§ 8. In the same manner as, for example, the numbers 7 and 8 constitute the fraction $\frac{7}{8}$, so Algebraical fraction. two quantities n and m constitute the algebraical fraction $\frac{n}{m}$; and all similar expressions are styled algebraical fractions.

§ 9. When an algebraical expression, either monomial or polynomial, is equal to another expression; for instance $a + b$ equal to $m + n$; we write Sign of equality and equations.

$$a + b = m + n$$

and read $a + b$ equal to $m + n$. The sign $=$ is called *sign of equality*. The expressions connected with this sign constitute an *equation*, and the first expression, as $a + b$, is called *first member*, and the last, $m + n$, second member of the equation.

§ 10. When an algebraical expression, for instance $2ab$, is greater than another, suppose $2m$, the inequality is expressed in either of the two following manners :

$$2ab > 2m$$

$$2m < 2ab$$

and accordingly the first is read $2ab$ *greater than* $2m$, and the second, $2m$ *less than* $2ab$. The angle or sign of inequality placed between the two unequal expressions must have the vertex toward the less of the compared terms.

§ 11. Two or more terms equal in every thing except the coefficient, are called *similar terms*; thus, for example $2a^2m$, $3a^2m$, a^2m are similar terms; and the terms mn , $\frac{1}{2}mn$, $5mn$, are likewise similar. Equal terms or expressions are perfectly identical, as for example, $2a^2$ and $2a^2$, or $m+n$ and $m+n$.

FIRST PART.

§ 12. WE will divide this preparatory treatise Division of the treatise. into two parts, like the larger. The first embraces simple algebraical operations, and the second, the principles or first elements of some algebraical theories.

The first part will contain three chapters: the first chapter, treating of addition and subtraction; the second, of multiplication and division; the third, of formation of powers and extraction of roots. Each chapter, moreover, will be divided into different articles, according as the operations are performed on monomial or polynomial, or fractional expressions.

CHAPTER I.

ADDITION AND SUBTRACTION.

ARTICLE I.

Addition and Subtraction of Monomials.

§ 13. ADDITION.—When simple terms or Addition, or sum of terms. monomials ought to be added, their *sum* is obtained by merely placing the sign + between them, (5;) so, for example, the addition or sum of the monomials ab , mn , $2t$, is represented by

$$ab + mn + 2t.$$

But sometimes the terms to be added are affected by different signs, or some of them are positive and some negative. To understand the meaning of posi- Positive and negative terms.

tive and negative terms or quantities, suppose the quantity b to be added to the quantity a , this a by the addition of b will become greater. But if, instead of adding b to a , we take c from a , then a will become smaller; b therefore increases and c diminishes a , or which is the same, b and c produce opposite effects, and for this reason b and c are termed *opposite* or *contrary* quantities or terms, and the one *positive* and the other *negative*. The meaning, therefore, of positive and negative terms is the same as that of opposite terms. Positive terms are generally marked with the sign plus, which for this reason is called also the *positive sign*; so for example: $+a$, $+2ab$ are considered as positive terms, although when no sign is placed before a term, the term is considered as positive. The sign $-$ or *negative sign* is always placed before negative terms.

Addition of terms
affected by different
signs.

§ 14. When the terms to be added are opposite terms, that is, some positive and some negative, as for example,

$$+2a, -b, +mn, -q,$$

the sum or addition is obtained by writing in succession all the terms, each with its own sign; so the sum of the preceding terms is

$$2a - b + mn - q$$

and supposing $a=7$, $b=5$, $m=1$, $n=2$, $q=8$, we will have $2a=14$, $mn=2$, and $2a-b=14-5=9$, $2a-b+mn=9+2=11$, and

$$2a - b + mn - q = 11 - 8 = 3.$$

Rule.

The general rule to be followed in addition of monomials is, consequently, this:—*Write in succession the terms to be added, each with its own sign.*

Addition of similar
terms, their
reduction.

§ 15. Among the terms to be added there may occur some similar terms, (11;) as for example, $2a$ and $3a$, the first of which signifies (4) that a is taken twice; the second, that it is taken three times; hence if

$2a$ is added to $3a$, we will have a taken 5 times, or $5a$. Similar terms, therefore, can be reduced to a single term, having for coefficient a number equivalent to all the coefficients of the added terms. If the similar terms are all positive or all negative, the coefficient of their sum will be equal to the sum of those coefficients. If some of the similar terms are positive and some negative, take then the sum of the positive coefficients, and then that of the negative ones; the difference of the two sums will be the coefficient of the sum of all the similar terms; and this coefficient will be positive if the sum of the positive coefficients is greater than that of the negative: otherwise, it will be negative, or equal to zero.)

¶16. Let us see some examples:

Examples.

(1.) Given terms $2a^2b$, $+3a^2b$, $+7a^2b$

Place the terms one below the other, as follows, and sum up the coefficients: we will have

$$\begin{array}{r} 2a^2b \\ 3a^2b \\ 7a^2b \\ \hline \text{Sum} = 2a^2b + 3a^2b + 7a^2b = 12a^2b \end{array}$$

(2.) Given terms $-3m^2n^2$, $-7m^2n^2$, $-12m^2n^2$.

We will have

$$\begin{array}{r} -3m^2n^2 \\ -7m^2n^2 \\ -12m^2n^2 \\ \hline -22m^2n^2 \end{array}$$

Hence the sum of

$$-3m^2n^2 - 7m^2n^2 - 12m^2n^2 = -22m^2n^2$$

(3.) Given terms ab^2c , $-2ab^2c$, $-3ab^2c$, $+5ab^2c$, $-7ab^2c$, $+9ab^2c$,

We will have

$$\begin{array}{r} ab^2c \\ 5ab^2c \\ 9ab^2c \\ \hline \text{Sum of positive terms} = 15ab^2c \end{array}$$

$$\begin{array}{rcl} \text{And} & & -2ab^2c \\ & & -3ab^2c \\ & & -7ab^2c \\ & & \hline \end{array}$$

Sum of negative terms = $-12ab^2c$

Hence the sum of all the terms is equal to $15ab^2c - 12ab^2c$.

Now as 12 positive expressions equal to ab^2c would increase the number of the 15 contained in the first term, so 12 negative expressions equal to ab^2c will reduce their number to only 3 positive, 3 being the difference between 15 and 12.

The sum, therefore, of the given terms is $+3ab^2c$, that is,

$$ab^2c - 2ab^2c - 3ab^2c + 5ab^2c - 7ab^2c + 9ab^2c = 3ab^2c.$$

(7.) Given terms,

$$ab^2c^3, -3ab^2c^3, +2ab^2c^3 - 5ab^2c^3.$$

$$ab^2c^3$$

$$2ab^2c^3$$

$$\text{Sum of positive terms} = 3ab^2c^3$$

$$-3ab^2c^3$$

$$-5ab^2c^3$$

$$\text{Sum of negative terms} = -8ab^2c^3$$

Now the difference of the coefficients 8 and 3 is 5; and since the negative sum is greater than the positive, the total sum will be negative, and we will have

$$ab^2c^3 - 3ab^2c^3 + 2ab^2c^3 - 5ab^2c^3 = -5ab^2c^3.$$

Reduction of similar terms.

This sum of similar terms, represented by only one equivalent term, is called *reduction of similar terms*. And since it is generally expedient to simplify operations and their results, so whenever similar terms occur among the given terms to be added, we first reduce the similar ones to a single term, and we add this term to the others. Hence in case of similar terms we observe the following

RULE.—*Reduce similar terms, and after reduction, write all the terms in order each with its own sign.*

§ 17. (5.) Given monomials.

$$\text{Examples. } 3a^4b^3, -mn, +2mn, -rs, -7a^4b^3, -5mn, +4a^4b^3.$$

In this example there are two sets of similar terms, and each set contains positive and negative terms. The positive terms of one set are $3a^2b^2, 4a^2b^2$, which reduced give $+7a^2b^2$; the negative term is $-7a^2b^2$, the first then of these two terms adds to the other monomials as much as the second takes from them; therefore, the addition of these two terms does not produce any change in the sum of the others. The two terms, therefore $+7a^2b^2, -7a^2b^2$ can be omitted in the

Terms mutually destroyed. Whenever two equal terms affected by opposite signs are added to one another, they are mutually destroyed, that is, both together are equal to zero.

The negative similar terms of the other set are $-mn$ and $-5mn$, which reduced give $-6mn$, and this term together with the other $+2mn$ gives again after reduction $-4mn$, hence the sum of the given monomials is

$$-4mn - rs.$$

(6.) Given $2ab, -mn, +r, -2mn, +3r, -4ab, +7mn, +2ab, -4mn, +r$.

(7.) Given $4f, +3cd - f, -2ab, +cd, +v, -3f, -3cd, +ab, -r, +2ab$.

(8.) Given $3ab, +4de, +2cd, +ab, -de, +3de, -ab, -3de, +ab, -3ab, +4ef$.

(9.) Given $a^2b, +3mnq, -f, -mnq, +3f, -4a^2b, +mnq, +f, +3a^2b, -2mnq, +a^2b, -2f$.

(10.) Given $4m^2r, +4fg, -m^2, -7m^2r, -3fg, +r^2, +6m^2r$.

(11.) Given $4d^2, +fd, -4ab^2, -6d^2, -lmq, +2ab^2, +2d^2, -g, +2ab^2$.

Answers:

(6.) Sum or $S = 5r$.

(7.) " $S = ab + cd$.

(8.) " $S = ab + 2cd + 3de + 4ef$.

(9.) " $S = a^2b + mnq + f$.

(10.) " $S = 3m^2r + fg - m^2 + r^2$.

(11.) " $S = fd - lmq - g$.

Numerical substitutions. § 18. Let us now make some numerical substitutions, and suppose $r = 8$ in the sum (6,) and, in the following (7,) make

$$a = 2, b = 3, c = 4, d = 5;$$

in the (8,)

$$a = 1, b = 1, c = 5, d = 3, e = 2, f = 1;$$

in the (9,)

$$a = 2, b = 3, m = 1, n = 4, q = 2, f = 1;$$

in the (10,)

$$m = 2, r = 1, f = 3, g = 4, t = 1;$$

in the (11,)

$$f = 10, d = 9, l = 8, m = 7, q = 6, g = 5;$$

we will find

$$(6.) S = 40$$

$$(7.) S = 26$$

$$(8.) S = 57$$

$$(9.) S = 21$$

$$(10.) S = 33$$

$$(11.) S = -251$$

Remark on the subject.

Numerical substitutions can be always made, because the quantities represented by algebraical symbols have their numerical values, although, as we remarked already, the same symbol can be used in different operations to represent different quantities of the same kind having different numerical values.

What subtraction is. § 19. SUBTRACTION.—To subtract a quantity b from another quantity a means to find the difference between the two quantities. Suppose, for example, $a = 10$ and $b = 7$, the difference between 10 and 7 is 3. Now if the numerical value of a quantity c is equal to 3, c is the difference of the terms a and b . But this difference is evidently obtained by taking b from a ; therefore, to subtract b from a means nothing else than to take from a a quantity equal to b ; which is represented (5) by $a - b = d$; d represents the difference.

§ 20. The difference is easily obtained when different cases to be considered. b is less than a ; if a , for instance, be equal to $5m$, and b to $2m$, we manifestly have

$$5m - 2m = 3m,$$

that is, $d = 3m$.

But if b is greater than a , and if a is negative and b positive, or b negative and a positive, or both negative, the difference is not equally evident.

Suppose b and a positive, and b greater than a .

First case. Taking a from b , that is

$$b - a,$$

— the difference is certainly positive. Hence, taking b from a , namely,

$$a - b,$$

the difference must be negative; because the first binomial ($b - a$) is exactly the opposite of the second; but

$$b - a = +d,$$

therefore,

$$a - b = -d.$$

The absolute value of d is the same in both cases, and indicates how much b is greater than a ; but in the second case it is a negative quantity. The same thing may be deduced in this manner:—Since d is the absolute difference between a and b , and b is supposed greater than a , we have

$$a + d = b,$$

and since the difference $d - d = 0$, we have also

$$a + d - d = a,$$

that is,

$$b - d = a.$$

— Hence, when (of two positive quantities) the greater is subtracted from the smaller, the difference is negative.

Let us subtract again b from a ; but let now a be a negative, and b a positive quantity; the difference in this case is

$$-a - b.$$

Observe, in fact, that if we add $b - b$ to $-a$, we have

$$-a = -a + b - b.$$

Therefore, if we take $+b$ from the second number of this equation, the difference will be the same as that arising from taking b from the first; but taking $+b$ from the second member, it remains $-a - b$. Hence,

When a positive quantity is subtracted from a negative one, the difference is equal to the negative sum of the two quantities. We say the negative sum, because $-a - b$ is the binomial opposite to $+a + b$, and $+a + b$ gives the positive sum of a and b .

Third case.

Suppose now a positive, and b negative, and subtract b from a ; the difference in this case is, therefore, to take b from a ; it is the same as to take it from $b - d$; but taking b from $b - d$, it remains $-d$; therefore, taking b from a , we will have likewise $-d$, that is

$$a - b = -d$$

the positive sum

$$a + b$$

of the terms; because

$$a = a + b - b;$$

hence, to take $-b$ from $a + b - b$ is the same as to take $-b$ from a ; but taking $-b$ from $a + b - b$, the remainder or difference is $a + b$; hence, subtracting $-b$ from a , the difference is $a + b$, that is, *whenever a negative quantity is subtracted from a positive one, the difference is equal to the positive sum of the two quantities.*

Fourth case.

The last case is that of both terms a and b negative. Taking b from a , the difference will be given by

$$-a + b.$$

Because again to take $-b$ from $-a$ is the same as to take it from $-a + b - b$; but taking $-b$ from this trinomial, it remains $-a + b$; therefore $-b$ taken from $-a$ gives $-a + b$.

Now if the absolute value of a is greater than that of b , the

- difference will be negative; and if a is less than b , the difference is positive. Hence, *when a negative term is subtracted from another negative the difference also is negative, if the subtracted term is less than the other; and if the same term is the greater, the difference then is positive.*

§ 21. We are now able to give this general

General rule. rule:—*When a term b is to be subtracted from another a , change the sign of b and join it to a , as for addition. The joined terms give the difference.*

We have seen in fact in the preceding cases, the sign of the term a remains always unvaried in the difference, and the sign of b becomes negative when the quantity to be subtracted is positive, and it becomes positive when the quantity to be subtracted is given negative.

Examples.	§ 22.	(1.)	From m	take $2n$
		(2.)	" $3m^2$	" $-m$
		(3.)	" ab	" d
		(4.)	" $-q^2$	" $-r$
		(5.)	" $2a^2b$	" a^2b
		(6.)	" $3m^2b$	" $4m^2b$
		(7.)	" $2abm$	" $-3abm$
		(8.)	" $-l^2f$	" $7l^2f$
		(9.)	" $7a^2b^2$	" $-5a^2b^2$
		(10.)	" $2cd^2$	" $-4cd^2$

Answers :

- (1.) $d = m - 2n$
- (2.) $d = 3m^2 + m$
- (3.) $d = ab - d$
- (4.) $d = -q^2 + r$
- (5.) $d = a^2b$
- (6.) $d = -m^2b$
- (7.) $d = 5abm$
- (8.) $d = -8l^2f$
- (9.) $d = 12a^2b^2$
- (10.) $d = 6cd^2$

ARTICLE II.

Addition and Subtraction of Polynomials.

Addition of polynomials, the same as that of monomials.

§ ADDITION.—The addition of polynomials does not differ from the addition of monomials, since to add one polynomial to another, signifies to add a number of monomials to another number of monomials; and to add together several polynomials, signifies to add together as many monomials as are contained in the given polynomials. The same rule, therefore, given for the addition of monomials is applicable to the case of polynomials, and it may be modified as follows:

Rule. *RULE.—Write the given polynomials one after another, arranging the similar terms in vertical columns, and make the reduction.*

Examples. § 24. (1.) Given polynomials :

$$\begin{aligned} p' &= a^2b - 3m + l \\ p'' &= m^2 - 2a^2b + 7m \\ p''' &= 4l - m + 3a^2b \end{aligned}$$

The same polynomials, with the similar terms arranged, will give

$$\begin{aligned} p' &= a^2b - 3m + l \\ p'' &= -2a^2b + 7m + m^2 \\ p''' &= 3a^2b - m + 4l \end{aligned}$$

$$\text{And } p' + p'' + p''' = 2a^2b + 3m + 5l + m^2$$

$p' + p'' + p'''$, or S , can be used to represent the sum of the given polynomials; in our case therefore we have

$$S = 3m + 5l + m^2 + 2a^2b.$$

(2.) Add together

$$\begin{aligned} p' &= 3a^2q - b^2l + 4a^2 - f \\ p'' &= 4f - 7a^2 \\ p''' &= 2b^2l - 4a^2q. \end{aligned}$$

(3.) Add together

$$p' = mn + m^2n^2 + 2ab$$

$$p'' = 3a^2b - 2m^2 + n.$$

(4.) Add together

$$p' = 3m^2r + 4m^2r^2 - 5mr^3$$

$$p'' = 2a^2 - 3bcd + m^2r$$

$$p''' = 4mr^2 - a^2 + 4bcd$$

$$p'' = mr^2 - bcd - 4m^2r.$$

(5.) Add together

$$p' = 13f^2l - 18g^2r^2 + qt$$

$$p'' = 7qt + f^2l + 3g^2r^2$$

$$p''' = 2ab - 9qt - 12f^2l.$$

Answers :

$$(2) S = b^2l - a^2q - 3a^2 + 3f$$

$$(3) S = mn + m^2n^2 + 2ab + 3a^2b - 2m^2 + n$$

$$(4) S = 4m^2r^2 + a^2$$

$$(5) S = 2f^2l - 15g^2r^2 - qt + 2ab.$$

Numerical substitutions can easily be made in all the foregoing expressions.

Polynomials considered as simple quantities.

§ 25. SUBTRACTION.—To subtract a polynomial B from another polynomial A, means, to find the difference of the two polynomials. Now, all that has been remarked with regard to the quantities a and b can be evidently applied to the polynomials A and B. Hence,

a polynomial B is subtracted from A, by adding to A the polynomial opposite to B, (21.)

Rule.

Now, we need not prove that the polynomial opposite to B contains the same terms, but with changed signs. So, for example, the polynomial

$$a^2b + mn - q$$

is opposite to

$$-a^2b - mn + q,$$

and if the former is represented by $+B$, the latter is equal to $-B$, and *vice versa*.

§ 26. But let us see some examples. (1.)

Examples.

Take the polynomial

$$B = 3a^2d - 4mn + g$$

from the polynomial

$$A = 7a^2d - 3mn + 2g;$$

D, or the required difference, is given by $A - B$, that is,

$$D = A - B.$$

Now, B or $+B$ is equal to the given polynomial; therefore,

$$-B = -3a^2d + 4mn - g;$$

hence, $A - B$ is equal to the following sum:

$$A = 7a^2d - 3mn + 2g$$

$$-B = -3a^2d + 4mn - g$$

$$\hline + 4a^2d + mn + g;$$

that is to say,

$$D = 4a^2d + mn + g.$$

(2.) Take $B = a + 2b + c + 5d$

from $A = 4a + 3b - 2c + 8d$.

(3.) Take $B = 7b^2l - 3b^2a^2 - 4bd + qrs$

from $A = -4bd + 2qrs - 2g + 8b^2l$.

(4.) Take $B = m^2 + 2amn - n^2 + t$

from $A = 2n^2 + tq - 3amn$.

(5.) Take $B = 2a^2b^2 - 3a^2b - 4ab^4$

from $A = a^4 - a^2b^2 + ab^4 - 3a^2b$.

Answers :

(2.) $D = 3a + b - 3c + 3d$.

(3.) $D = b^2l + qrs - 2g + 3b^2a^2$.

(4.) $D = 3n^2 + tq - 5amn - m^2 - t$.

(5.) $D = a^4 - 3a^2b^2 + 5ab^4$.

ARTICLE III.

Addition and Subtraction of Fractional Terms.

Algebraic fractions to be treated like numerical fractions. § 17. WE have seen (7) that the expression $\frac{n}{m}$ is considered in algebra a fraction, or fractional expression, no less than in arithmetic $\frac{7}{8}$, for example, or $\frac{9}{8}$ are fractions, or numbers having a fractional form.

Now n and m , and generally any quantity, has its numerical value, and therefore algebraic fractional terms are to be treated like numbers of a fractional form.

But the signification of a simple numerical fraction, for example $\frac{7}{8}$, is this, that seven parts are taken out of the eight, into which the unity is conceived to be divided, and 7, which indicates the number of the parts taken, is called *numerator*, and 8, which shows into how many equal parts unity is supposed to be divided, is termed *denominator*. All this is to be manifestly applied to the numerical value of algebraical fractions; because, considering in the fraction $\frac{n}{m}$ the numerical value of its terms, the signification of $\frac{n}{m}$ is the same as that of a fractional number.

Now a numerical fraction having its numerator equal to the denominator is equal to unity, for instance, $\frac{8}{8} = 1$, $\frac{12}{12} = 1$, &c., because the parts into which unity is supposed to be divided, are all taken: hence, also an algebraical fraction $\frac{n}{n}$ has its numerical value equal to unity.

From what we are about to say in the following article, we will see that a number or quantity multiplied by unity is equal to itself; hence it follows that any quantity multiplied

by $\frac{n}{n}$ remains unvaried; thus, a multiplied by $\frac{n}{n}$ is equal to a ; and $\frac{a}{b}$ multiplied by $\frac{m}{m}$ is equal to $\frac{a}{b}$, &c.

Observe also that a numerical fraction is multiplied by another numerical fraction, by taking the product of the numerators and dividing it by the product of the denominators; for instance, $\frac{7}{8}$ is multiplied by $\frac{3}{4}$, by taking $\frac{7}{8} \cdot \frac{3}{4}$; the same must be done with algebraical fractions. We will better see in the following articles the process of this operation. Applying, meanwhile, the arithmetic rule to algebraical fractions, let us admit that $\frac{m}{n} \cdot \frac{r}{5} = \frac{m \cdot r}{n \cdot 5}$ and $\frac{m}{n} = \frac{m}{n} \cdot \frac{r}{r} = \frac{m \cdot r}{n \cdot r}$.

Fractions reduced
to the same deno-
minator.

§ 28. We may now proceed to reduce different fractions to the same denominator without changing their values. Let

$$\frac{a}{b}, \quad \frac{c}{d}, \quad \frac{e}{r}$$

be the given fractions.

From what has been just remarked, it follows that

$$\frac{a}{b} = \frac{a}{b} \cdot \frac{d}{d} = \frac{ad}{bd} = \frac{ad}{bd} \cdot \frac{r}{r} = \frac{adr}{bdr},$$

and

$$\frac{c}{d} = \frac{c}{d} \cdot \frac{b}{b} = \frac{cb}{db} = \frac{cb}{db} \cdot \frac{r}{r} = \frac{cbr}{dbr}$$

$$\frac{e}{r} = \frac{e}{r} \cdot \frac{b}{b} = \frac{eb}{rb} = \frac{eb}{rb} \cdot \frac{d}{d} = \frac{ebd}{rbd}.$$

Now whatever be the numerical value of the terms b, d, r ; b multiplied by d , and by r , gives the same result in whatsoever order the terms may be taken; that is, we have

$$bdr = dbr = rbd.$$

Suppose, in fact, for example $b = 2, d = 3, r = 4$, then

$$bdr = 2 \cdot 3 \cdot 4 = 24$$

$$dbr = 3 \cdot 2 \cdot 4 = 24$$

$$rbd = 4 \cdot 2 \cdot 3 = 24.$$

Hence, taking the first member and the last of the preceding equations, and changing in the second dbr into bdr , and in the third rbd into bdr , we have

$$\frac{a}{b} = \frac{adr}{bdr}$$

$$\frac{c}{d} = \frac{cbr}{bdr}$$

$$\frac{e}{r} = \frac{ebd}{bdr}$$

In this manner the given fractions are transformed into other fractions, having all the same denominator; but $\frac{a}{b}$ is transformed into $\frac{adr}{bdr}$ by multiplying a and b by both the denominators of the other two fractions; and $\frac{c}{d}$ is changed into $\frac{cbr}{bdr}$ by multiplying c and d by the denominators of the other two fractions; and so likewise the last fraction is changed. Hence the

Rule and examples.

RULE.—*Fractions are reduced to the same denominator by multiplying the numerator and denominator of each by the denominator of all the others.*

EXAMPLES.—Reduce to the same denominator the following fractions :

Given (1.) $\frac{a}{b}, \frac{c}{d}, \frac{m}{n}, \frac{r}{s}, \frac{p}{q}$

(2.) $\frac{a^2}{b^2}, \frac{m}{n^2}, \frac{q^2}{s^2}$

(3.) $\frac{lm}{rg}, \frac{tf}{sq}, \frac{np}{vu}$

(4.) $\frac{a}{b^2}, \frac{c}{a^2}, \frac{d}{mn}$

(5.) $\frac{rad^2}{stv}, \frac{mqs}{gf}, \frac{fabg^2}{d}$

Answers:

- (1.) $\frac{adnsq}{bdnsq}, \frac{cbnsq}{bdnsq}, \frac{mbdsq}{bdnsq}, \frac{rbdnq}{bdnsq}, \frac{pbdns}{bdnsq}$
- (2.) $\frac{a^2n^2s^4}{b^2n^2s^4}, \frac{mb^2s^4}{b^2n^2s^4}, \frac{q^2b^2n^2}{b^2n^2s^4}$
- (3.) $\frac{lmsqvu}{rgsqvu}, \frac{tfrgvu}{rgsqvu}, \frac{nprgsq}{rgsqvu}$
- (4.) $\frac{a^3mn}{b^3a^3mn}, \frac{cb^3mn}{b^3a^3mn}, \frac{db^3a^3}{b^3a^3mn}$ (See n. 4.)
- (5.) $\frac{rd^4gf}{stvqfd}, \frac{mqs^2tvd}{stvqfd}, \frac{f^2abg^2stv}{stvqfd}$

Reduction of fraction to lower terms.

§ 30. In the same manner as a fraction without changing its value can have its numerator and denominator multiplied by any other quantity, so if the numerator and denominator of a given fraction are multiplied by the same quantity, the fraction may be reduced to lower terms, as follows:

Suppose $\frac{mn}{rn}$ to be the given fraction; we may reduce this fraction to $\frac{m}{r}$; because $\frac{m}{r} = \frac{m}{r} \cdot \frac{n}{n} = \frac{mn}{rn}$; hence the rule,

Rule. *When the numerator and denominator have a common factor, the fraction is reduced to similar terms, by the omission of that factor.*

So, for example, the fractions

$$\frac{am}{bm}, \frac{rs}{br}, \frac{tfg}{gt}, \frac{m^2b}{4m^2l}, \frac{7ab}{3be}$$

are easily reduced to the simple forms

$$\frac{a}{b}, \frac{s}{b}, \frac{f}{q}, \frac{b}{4l}, \frac{7a}{3e}$$

And since b^3 , for instance, is equal to $b \cdot b \cdot b$, and $b^4 = b \cdot b \cdot b \cdot b$, and consequently the fraction

$$\frac{m^2b^3}{n^2b^4} = \frac{m^2bbb}{n^2bbbb} = \frac{m^2}{n^2b}$$

so when the same quantity is to be found in the numerator and

denominator, with different exponents, the reduction is then performed by suppressing the quantity with the lower exponent, and diminishing the higher exponent by as many units as there are in the smaller; thus for example, the fractions

$$\frac{am^5}{3m^2b}, \frac{n^2m^2b}{n^3b}, \frac{l^2r^2s^4}{l^4r^2s^2}, \frac{c^3d^5}{c^4d^4}$$

are easily reduced to a simple form, as follows :

$$\frac{am^3}{3b}, \frac{m^2}{n}, \frac{s^2}{l^2}, \frac{d}{c}.$$

§ 31. ADDITION.—It is now easy for us to add together terms having a fractional form. Observe that the sum of the following numerical fractions,

$$\frac{3}{8}, \frac{4}{8}, \frac{5}{8},$$

having all the same denominator, is obtained by adding together the numerators, and giving to the sum the same 8 for denominator; because this sum signifies, that, of the eight parts in which unity, or several units are conceived to be divided, we take as many as there are units, in all the numerators of the given fractions. So in the proposed example, we take three of such parts, and four more, and then five more; hence the sum of all is manifestly given by $1\frac{2}{8}$. Now considering the numerical value of algebraical fractions, the sum must be obtained in the same manner; but the sum of the numerators can only be taken when all the denominators are equal; therefore in the addition of algebraical fractions, the first rule is

Reduce the terms to the same denominator.

Rule 1.

Then without having any consideration for the denominator, we make the addition of numerators, and consequently the operation becomes the same as that considered in the preceding articles; so that the next rule can be expressed as follows :

Take the sum of the numerator, and divide this

Rule. *sum by the common denominator.*

Examples. § 32. (1.) Given fractions, $\frac{3m^2}{n}, -\frac{n}{m}, +\frac{b}{d}$, the same terms, reduced to the same denominator, become

$$\frac{3m^2d}{nmd}, -\frac{n^2d}{nmd}, \frac{mb}{nmd};$$

and the sum of the numerators is

$$3m^2d - n^2d + nmb;$$

therefore the sum of the given fraction is

$$S = \frac{3m^2d - n^2d + nmb}{nmd}.$$

$$(2.) \text{ Given } \frac{m}{n^2}, \frac{n}{m^2}, \frac{q}{rs}, \frac{f}{g}.$$

$$(3.) \text{ " } \frac{ab}{q}, -\frac{rs}{ab}, +\frac{q^2}{m}, -\frac{m}{g}.$$

$$(4.) \text{ " } \frac{l^2}{r}, \frac{f}{q^2}, -\frac{a}{b}.$$

Answers :

$$(2.) S = \frac{m^2rsq + n^2rsq + qn^2m^2g + fn^2m^2rs}{n^2m^2rsq}.$$

$$(3.) S = \frac{a^2b^2mg - rsqmg + q^2abg - m^2qab}{qabmg}.$$

$$(4.) S = \frac{l^2q^2 + frb - arq^2}{rq^2b}.$$

Sometimes the reduction of terms to the same denomination can be more expeditious. Suppose, for instance, that the terms, $\frac{mn}{bq}, -\frac{a}{q^2}, +\frac{r}{b}$, are to be added.

$$(5.) \text{ Given } \frac{mn}{bq}, -\frac{a}{q^2}, +\frac{r}{b}.$$

In this case, without multiplying the numerators and denominators of each fraction by all the other denominators, it is enough to multiply the numerator and denominator of the first fraction by q , the numerator and denominator of the second by b , and the numerator and denominator of the last by q^2 , and so we will have

$$(5.) \quad S = \frac{mnq - ab + rq^2}{bq^2}.$$

So likewise from

$$(6.) \quad \text{Given } \frac{2s^2t}{mnq} - \frac{lm}{n^2q} + \frac{u}{q^2},$$

$$(7.) \quad \text{“} \quad \frac{ab}{cd} \frac{rs}{q} - \frac{l}{d^2} + \frac{f}{cq},$$

$$(8.) \quad \text{“} \quad \frac{a^2}{3m} - \frac{m}{n^2} \frac{bd}{mn},$$

we have

$$(6.) \quad S = \frac{2s^2tnq - lm^2q + umn^2}{mn^2q^2}.$$

$$(7.) \quad S = \frac{abqd + rscd^2 - lcg + fd^2}{cqd^2}.$$

$$(8.) \quad S = \frac{a^2n^2 - 3m^2 + 3bdn^2}{3mn^2}.$$

Other examples of terms given with the same denominator :

$$(9.) \quad \text{Given } \frac{m^2b}{n^2c} - \frac{a^2}{n^2c} + \frac{2m^2b}{n^2c} + \frac{3a^2}{n^2c}.$$

$$(10.) \quad \text{“} \quad \frac{a^2b}{qt} - \frac{lm}{qt} - \frac{3a^2b}{qt} + \frac{2lm}{qt} + \frac{2a^2b}{qt}.$$

$$(11.) \quad \text{“} \quad \frac{4a^2b}{rs^2} - \frac{2a^2}{rs^2} + \frac{lg^2}{rs^2} - \frac{3a^2b}{rs^2} + \frac{a^2}{rs^2} - \frac{2lg^2}{rs^2} \\ - \frac{a^2b}{rs^2} + \frac{3a^2}{rs^2}.$$

Answers :

$$(9.) \quad S = \frac{3m^2b + 2a^2}{n^2c}.$$

$$(10.) \quad S = \frac{lm}{qt}.$$

$$(11.) \quad S = \frac{2a^2 - lg^2}{rs^2},$$

§ 33. SUBTRACTION.—Numerical fractions are easily taken from one another when they have the same denominator; because, were the fraction $\frac{2}{3}$ to

be taken from $\frac{5}{8}$, the difference, $5 - 3 = 2$, divided by 8, would be the difference of the two fractions, since 3 of the 8 parts of unity being taken from 5, there can remain but 2; hence $\frac{2}{8}$ is the difference of the given fractions. Let now the same be said of the difference of algebraical fractions, or of that of their numerical values. Hence the rule,

Rule. Reduce the terms to the same denomination, and by this denominator divide the difference of the numerators.

Now the difference of the numerators is obtained in the same manner as that of the simple terms; applying therefore the general rule (21) to the present case, we will obtain the differences of the following terms:

Examples.	(1.)	Take $\frac{a}{b}$	from	$\frac{m}{n}$
	(2.)	" $\frac{a^2m}{c}$	"	$-\frac{c^2}{m}$
	(3.)	" $\frac{lm}{s}$	"	$\frac{tq}{g}$
	(4.)	" $-\frac{rt}{f^2}$	"	$\frac{q^2r}{s^2}$
	(5.)	" $-\frac{sg^2}{lm}$	"	$-\frac{bd}{f^2}$
	(6.)	" $\frac{a^2f}{lt}$	"	$\frac{f^2}{l^2}$
	(7.)	" $-\frac{m^2n^2}{ab^2}$	"	$-\frac{m}{a^2b}$
	(8.)	" $\frac{2ab}{m}$	"	$\frac{3a}{m}$
	(9.)	" $-\frac{2m^2c}{n^2}$	"	$\frac{3m^2c}{n^2}$
	(10.)	" $\frac{b^2q}{mn}$	"	$-\frac{3b^2q}{mn}$

Answers :

$$(1.) \quad d = \frac{mb - an}{bn}.$$

$$(2.) \quad d = \frac{-c^2 - a^2m^2}{cm}.$$

$$(3.) \quad d = \frac{tqs - lmg}{sg}.$$

$$(4.) \quad d = \frac{q^2rf^2 + rts^2}{f^2s^2}.$$

$$(5.) \quad d = \frac{-bdlm + sg^2f^2}{lmf^2}.$$

$$(6.) \quad d = \frac{f^2t - a^2fl}{l^2t}.$$

$$(7.) \quad d = \frac{-mb + m^2n^2a}{a^2b^2}.$$

$$(8.) \quad d = \frac{3a - 2ab}{m}.$$

$$(9.) \quad d = \frac{5m^2c}{n^3}.$$

$$(10.) \quad d = -\frac{4b^2q}{mn}.$$

CHAPTER II.

MULTIPLICATION AND DIVISION.

Remarks. § 34. IN the same manner as subtraction, is the operation opposite to addition, so division is opposite to multiplication; hence, since multiplication is the addition of the same quantity to itself a certain number of times, so division is equivalent to the subtraction of the same quantity from another likewise a certain number of times. Multiplication of algebraical terms or quantities is the same as the

multiplication of the numerical value of the same quantities, and so likewise for division.

Numerical multiplication. Now a number, for example 7, is said to be multiplied by another, for instance 5, when 7 is added to itself five times; that is,

$$7 \cdot 5 = 7 + 7 + 7 + 7 + 7 = 35.$$

And the number 35 is said to be divided by 5, when we find out how many times 5 may be taken from 35, or, which is the same, how many times 5 enters into 35.

These principles applied to quantities give us a sufficient idea of the operations before us.

ARTICLE I.

Multiplication and Division of Monomials.

Definitions and different cases. **MULTIPLICATION.**—When a term a is multiplied by another term b , the first is called *multiplicand*, and the latter *multiplier*; both together are termed *factors*; the result of the operation is termed *product*.

With regard to the factors, we can distinguish four different cases. Both factors may be positive, the multiplicand can be positive, and the multiplier, negative; the multiplicand may be negative, and the multiplier positive; and finally, both negative. Let us now see what must be the sign to be given to the product in these different cases.

Signs of the product. §36. The multiplier b may be called the index of the operation to be performed on the the multiplicand a ; so, for instance, if the numerical value of b is 3, b indicates that a is to be added three times to itself; now when a is positive, this quantity added to itself three, four, or any number of times, gives necessarily a positive

sum : calling then p the product, in the case of both factors, positive, we have

$$+a \times +b = +p;$$

but if a is negative, and b positive as before, then b indicates with regard to $-a$, the same operation as with regard to $+a$, it namely indicates that $-a$ is to be added to itself three times; but $-a$ added any number of times to itself gives evidently a negative sum, therefore in the case of the multiplicand negative, and the multiplier positive, we have

$$-a \times +b = -p.$$

Let us suppose again a positive, but b negative: the multiplier $-b$ indicates that a is to be added to itself three times, but since $-b$ is opposite to $+b$, the addition of a must be made in a manner opposite to the first, which cannot be done except by adding a to itself with the sign opposite to its own; and since in the present case, a is supposed positive, so we have an addition of negative terms, which gives a negative sum, and consequently

$$+a \times -b = -p.$$

In like manner, since $-a$, multiplied by $+b$, gives for product the addition of $-a$ repeated a certain number of times; so $-a$, multiplied by $-b$, will give for product the addition of $-a$ the same number of times, but taken with opposite sign changed namely into $+a$, and therefore

$$-a \times -b = +p.$$

Hence, when the signs of the factors are both positive, or both negative, the product is positive; when the sign of one of the factors is positive, and the sign of the other is negative, the product is negative. All this is simply expressed by the rule:

Like signs give a positive, unlike signs give a negative product.

Rule of signs.
Sign to be given
to the product of
several factors.

§ 36. It is now easy to determine what sign is to be given to the product effected by several

factors, a, b, c , &c. When all the factors are positive, the product must necessarily be positive; because the product of the first two terms is certainly positive, that is

$$a \cdot b = +p;$$

but the product of the first three terms also is positive, because $p \cdot c = +p'$, and since $p = a \cdot b$,

$$a \cdot b \cdot c = +p';$$

in like manner we prove that four and five, and any number of positive factors, give a positive product.

But if among the factors there is one, and only one negative, then the product of all must be negative; because, let the negative factor be either the first or the second, then the first product will be negative; but multiplying $-p$ by the next factor c , a positive term, the product p' also is negative; and so likewise p' , multiplied by the next factor, gives a negative product, and so on till the last factor. But if the negative factor is neither the first nor the second, but in any place among the others, then the product of all the factors preceeding is certainly positive, and this product multiplied by the negative term produces a negative result; which, multiplied by the following term, gives again a negative product, and so on until the last term as before. If the negative factors are two, then the product is positive, because the negative product effected by the first negative sign is changed into positive by the second, and so will remain until the last term, and this whatever be the arrangement of the terms. Hence, when among the factors there are only two negative, the product is positive, as if all the factors should be positive. Therefore, when the negative factors are three or four, they will affect the sign of the product, as when they are only one and only two, or when they are only one and none; therefore, three negative terms give a negative, and four negative terms give a positive product. The same, and in the same manner, is

demonstrated of five and six terms; and generally when the number of negative terms is an even number, the product is positive; when the number of negative terms is an odd number, the product is negative; hence the general rule:

Rule. Whenever the number of negative factors is an odd number, the product is negative: otherwise, positive.

For example, the factors,

$$(1.) \quad a, b, c, d, e, f, g, h,$$

$$(2.) \quad a, b, -c, d, -e, f, -g, h,$$

$$(4.) \quad a, -b, c, -d, e, -f, g, -h,$$

$$(5.) \quad -a, -b, -c, -d, -e, -f, -g, -h,$$

given (7) the products

$$(1.) \quad p = abcdefgh.$$

$$(2.) \quad p = -abcdefgh.$$

$$(3.) \quad p = abcdefgh.$$

$$(4.) \quad p = abcdefgh.$$

Examples. The sign to be given to the product being thus determined for all cases, let us proceed to see some

examples of compound monomials, about which observe, first, that when two or more monomials have a coefficient, the coefficient may be multiplied separately; for example, $2ab$ multiplied by $3mn$ would give $3 \cdot 2 \cdot abmn$, or $6abmn$. Again, if the same term occurs repeatedly in different factors, it is enough to write it only once in the product with an exponent equal to the sum of the partial exponents; for instance, the product of a^2b and $2a$ and a^3m is (4) $2a^{(2+1+3)}m$, that is $2a^6m$; hence, besides the rule of signs, another rule is given for the multiplication of monomials:

Rule. Multiply the coefficients separately, and when the symbol occurs in different factors, write it only once with the exponent equal to the sum of the partial exponents.

Given factors:

$$(1.) \quad a^2bm, 3mn, 4n^2b.$$

$$(2.) \quad a^3c^2d, -mdc.$$

- (3.) $-fbl^2, -3gl.$
 (4.) $-3s^4md, 4dt.$
 (5.) $7m^3n^2, -5ab, +3a^2l, -4mb.$
 (6.) $13g^3rs, +12rs, -3abd, +2bc.$
 (7.) $m^2n, +3n^2b, -5b^3d, -d^2.$
 (8.) $-4a^2b, 5a^2b^2, -6lmn.$
 (9.) $-3rs, -4tq, -2rstq, -2s^2q^2.$
 (10.) $2g^4h, 3hs, -4sg, -5h^2k, +6g^2s^2k^2.$

Answers :

- (1.) $p = 12a^2b^2m^2n^2.$
 (2.) $p = -a^2c^2d^2m.$
 (3.) $p = 3fbgl^2.$
 (4.) $p = -12s^4d^2mt.$
 (5.) $p = 420m^4a^2n^2l^2l.$
 (6.) $p = -936g^2r^2s^2b^2acd.$
 (7.) $p = 15m^2n^4b^4d^2.$
 (8.) $p = 120a^{11}b^4lmn.$
 (9.) $p = 48r^2s^4tq^4.$
 (10.) $p = 720g^7h^5s^4k^3.$

§ 39. DIVISION.—To divide a quantity a by a quantity b , means to find another quantity q , which we call *quotient*, and which multiplied by b , or the divisor, ought to give the *dividend* a for product; therefore the dividend is considered as the product of two factors, one of which is the divisor, and the other is the quotient to be found.

Hence, we see why the quotient of a divided by b
 Corollary. is represented by the fractional expression $\frac{a}{b}$, (7, 8,)

since $\frac{a}{b}$ multiplied by b gives (as we will see hereafter) a for product.

§ 40. We infer, moreover, the rules of the
 Sign of quotient, different cases. signs to be given to the quotient. Suppose, first,

both terms, dividend and divisor, positive; then the quotient also must be positive, because the quotient multiplied by the divisor, that is by a positive term, ought to give a positive product equal to the dividend, which cannot be obtained unless q , together with b , the divisor, are affected by the same sign; hence,

$$\frac{+a}{+b} = +q.$$

Suppose again the divisor b positive, but the dividend negative; the quotient in this case must be negative, because $+b$, multiplied by q , ought to give the negative product $-a$, which cannot be obtained except with q negative, when b is positive; hence,

$$\frac{-a}{+b} = -q.$$

Suppose now b negative, and a positive; in this case the product of $-b$ multiplied by q must give the positive product $+a$; but this cannot be obtained unless both factors b and q are either both positive, or both negative; therefore, with b negative, q also must be negative; therefore,

$$\frac{+a}{-b} = -q.$$

Let finally both terms, divisor and dividend be negative; in this case $-b$, multiplied by q , ought to give a negative product; hence, q must be positive; therefore,

$$\frac{-a}{-b} = +q.$$

The same rule of sign therefore given (36) for the sign of the product holds good even for the sign of the quotient, that is,

Rule. *Like signs give a positive, and unlike signs give a negative quotient.*

Hence the terms,

- (1.) $+m$ divided by $+n$,
- (2.) $+r$ " " $-s$.

(3.) $-t$ divided by $+p$,

(4.) $-g$ " $-k$,

give the quotients,

$$(1.) \quad \frac{m}{n} = +q.$$

$$(2.) \quad \frac{r}{-s} = -q.$$

$$(3.) \quad \frac{-t}{p} = -q.$$

$$(4.) \quad \frac{-g}{-k} = +q.$$

§ 41. We observed (7) that both expressions
Examples. $a : b$, and $\frac{a}{b}$, denote the quotient of a divided by b ;

and since $\frac{a}{b}$ may be considered like any fractional expression : when a and b are compound monomials, and contain some common factor, this may be omitted without changing the value of the quotient: let, for instance, m^2b be the dividend, and ab the divisor, the quotient is $\frac{m^2b}{ab}$; but we remarked already (27 and 31) that $\frac{m^2b}{ab} = \frac{m^2}{a}$.

If the monomial have coefficients, these may be divided separately, or reduced to simple elements, like simple numerical fractions; hence the rule:

Rule. Divide the coefficients separately, and reduce the fractional expression to lower terms.

Given terms :

(1.) Divide $4a^2b$ by $5m^2b$.

(2.) " $2fm^2r^3$ " $7mr$.

(3.) " $3p^2sl$ " $18p^2s^2l^2$.

(4.) " $5g^2h^3$ " $-6g^2h^5$.

(5.) " $-7k^4tsl$ " $3kt^2mn$.

(6.) " $-s^2r^3b$ " $-5gl$.

- (7.) Divide $-8t^2r^2s^4$ by $2tr^2s^5$
 (8.) " $lmnr^2$ " $-5l^2m^2sg$.
 (9.) " $13s^2tr$ " $26b^2c^3$.
 (10.) " $26gkh^3$ " $-2g^2khtv$.

Answers:

- (1.) $q = \frac{4}{5} \frac{a^2}{m^2} = \frac{4a^2}{5m^2}$.
 (2.) $q = \frac{3 \cdot mr^2}{1} = 3mr^2$.
 (3.) $q = \frac{1}{6} \cdot \frac{1}{psl} = \frac{1}{6psl}$.
 (4.) $q = -\frac{5g^2}{6h^2}$.
 (5.) $q = -\frac{7k^2sl}{3tmn}$.
 (6.) $q = \frac{s^2r^2b}{5gl}$.
 (7.) $q = -\frac{4t^2r^4}{s}$.
 (8.) $q = -\frac{nt^2}{5lm^2sg}$.
 (9.) $q = \frac{s^2tr}{2b^2c^3}$.
 (10.) $q = -\frac{13h^2}{gtv}$.

ARTICLE II.

Multiplication and Division of Polynomials.

Object of the operation. § 42. MULTIPLICATION.—A polynomial A is multiplied by another polynomial B, taking, by way of addition, A in that manner and as many times as B indicates. P, or the intended result of the operation, is termed

product; and the factors A and B multiplicand and multiplier, as in the case of simple monomials.

The product is obtained by multiplying A by each of the terms of B, and summing up all these partial products. Now since the same terms, both in number and form, are obtained by multiplying A by each term of B, or B by each term of A; so the same sum or product is effected, taking either of the polynomials for multiplicand or multiplier; hence the rule:

Take the sum of the products A by each of terms of B, or vice versa.

For example, suppose $B = m + n - r$, and $A = c + d$, the product P will be

$$A \cdot m + An - Ar;$$

$$\text{Or } B \cdot c + B \cdot d, \quad \text{both equal.}$$

$$\text{In fact, } Am = (c + d)m = cm + dm$$

$$An = (c + d)n = cn + dn$$

$$-Ar = -(c + d)r = -cr - dr,$$

$$\text{and } Bc = (m + n - r)c = cm + cn - cr,$$

$$Bd = (m + n - r)d = dm + dn - dr;$$

in both cases the same terms, and consequently the same sum, or product.

It happens sometimes, that similar terms occur in the partial products, and then to render easier the addition, it is expedient to write these terms one after another in the same vertical column; which is easily obtained, if, before commencing the multiplication, the factors A and B be arranged according to the powers of the same letter, when it can be done.

Polynomials arranged. § 43. Let A, for instance, be equal to $4a^2b - ab^3 + 3a^2b^2$, and $B = 2ab + a^2 + b^2$. The first is arranged according to the powers of a, by writing

$$4a^2b + 3a^2b^2 - ab^3;$$

where the highest power is in the first place, the next highest

in the second, &c., and diminishes by degrees until the last term. In like manner, the polynomial B is arranged according to the powers of the same letter a , by writing,

$$a^3 + 2ab + b^3.$$

The arrangement could be made also, in the inverse order. Let us propose some examples. Arrange the

Given polynomials :

Examples.

$$(1.) \quad 3m^2n + 3mn^2 + 5n^3 - 6m^3.$$

$$(2.) \quad 7ab^3 - 5b^4 + 3a^2b^2 - a^3b - a^4.$$

$$(3.) \quad 7g^5l - l^3 + 3g^3l^2 - 4g^2l^4 - 5g^4l^2 - 9gl^5 + g^6.$$

Answers :

$$(1.) \quad -6m^3 + 3m^2n + 3mn^2 + 5n^3.$$

Writing the same terms in the inverted order, we would have the same polynomials arranged according to the increasing powers of m ; commonly, however, polynomials are arranged according to the decreasing powers of the same letter. The same polynomial (1) can be arranged according to the decreasing powers of n .

$$(2.) \quad -5b^4 + 7ab^3 + 3a^2b^2 - a^3b - a^4.$$

$$(3.) \quad g^6 + 7g^5l + 5g^4l^2 + 3g^3l^3 - 4g^2l^4 - 9gl^5 - l^6.$$

Examples of
multiplication.

§ 44. Multiply

$$(1.) \quad \begin{cases} A = a^3 + 3a^2b + 3ab^2 + b^3 \\ \text{by} \\ B = a^2 + 2ab + b^2. \end{cases}$$

$$(2.) \quad \begin{cases} A = 6m^2n^3 + 3m^4n - 7mn^4 - 5m^2n^2 \\ \text{by} \\ B = -3mn + n^3 + 2m^2. \end{cases}$$

$$(3.) \quad \begin{cases} A = 4a^3 - 8ab^2 - 5a^2b + 2b^3 \\ \text{by} \\ B = 2a^2 - 4b^2 - 3ab. \end{cases}$$

$$(4.) \quad \begin{cases} A = 7m^3l - 32m^2 + bc \\ \text{by} \\ B = 3m - l + g. \end{cases}$$

$$(5.) \quad \begin{cases} A = a^2 + b^2 + c^2 - ab - ac - bc \\ \text{by} \\ B = a + b + c. \end{cases}$$

$$(6.) \quad \begin{cases} A = m^2 + mn + n^2 \\ \text{by} \\ B = m^2 - mn + n^2. \end{cases}$$

Answers: (1.) The polynomials of the first example are both already arranged according to the decreasing powers of a .

Call p' , p'' , p''' the partial products of A by the first, second, and third terms of B and P , or $p' + p'' + p'''$, the total product. This will be easily obtained in the following manner:

$$\begin{aligned} A \cdot B &= \underline{(a^2 + 3a^2b + 3ab^2 + b^2)(a^2 + 2ab + b^2)} \\ p' &= \dots a^5 + 3a^4b + 3a^3b^2 + a^2b^3 \\ p'' &= \dots 2a^4b + 6a^3b^2 + 6a^2b^3 + 2ab^4 \\ p''' &= \dots \dots \dots a^3b^3 + 3a^2b^3 + 3ab^4 + b^5 \\ P &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \end{aligned}$$

The same process of operation is observed whenever it is possible. When no similar terms occur, the product P is then obtained by writing in succession all the terms of the partial products.

In the second example, arrange the terms of the polynomials according to the powers of m , and you will have

$$(2.) \quad P = 6m^4n - 19m^3n^2 + 30m^2n^3 - 37m^2n^4 + 27m^2n^5 - 7mn^6.$$

And from the other given polynomials you will find

$$(3.) \quad P = 8a^5 - 22a^4b - 17a^3b^2 + 48a^2b^3 + 26ab^4 - 8b^5.$$

$$(4.) \quad P = 21m^4l - 96m^3 + 3bcm - 7m^2l^2 + 32m^2l - bcl + 7m^2lq - 32m^2q + beq.$$

$$(5.) \quad P = a^3 + b^3 + c^3 - 3abc.$$

$$(6.) \quad P = m^4 + m^2n^2 + n^4.$$

§ 45. The preceding examples have both
Common factors. factors polynomials. It is easy to see that the case of one of the factors being monomial is contained in

this, and exemplified in obtaining the partial products. The case, however, of one of the factors polynomial, and the other monomial, deserves to be noticed on account of the so-called *common factor* produced by the monomial.

$$\begin{aligned}\text{Let } A &= m^2n + rs + t, \\ \text{and } B &= a,\end{aligned}$$

we will have

$$P = am^2n + ars + at;$$

a is a factor of each term of P , that is a common factor: now

$$\begin{aligned}P &= B \cdot A \\ &= a(m^2n + rs + t);\end{aligned}$$

therefore the common factor of different terms, instead of being joined to each term, may be written only once as multiplier or multiplicand of the same terms. So we will have

$$\begin{aligned}m^2rs + m^2gq + 3m^2f &= m^2(rs + gq + 3f) \\ 7ng^2d + \frac{1}{2}n^2qd + n^3 &= n(7g^2d + \frac{1}{2}nqd + n^2) \\ mn^2 + 2nd + gl + tg &= n(mn + 2d) + g(l + t) \\ fr^2 + mn + m^2 + t &= fr^2 + t + m(n + m).\end{aligned}$$

Observe that a binomial, nay a polynomial also, is sometimes used as a simple common factor.

§ 46. DIVISION.—The object of the division of a polynomial by another polynomial is the same as that of the division of monomials; hence, when the polynomial A is given to be divided by the polynomial B , the former A is to be considered as the product of B by Q , the quotient to be found. Here also, A is termed dividend, and B divisor.

The polynomials ought to be arranged. In the division of one polynomial by another, it is more expedient than in multiplication to have the polynomials arranged according to the powers of the same letter.

We may easily see in the preceding examples of multiplication, that when both factors are arranged according to the

power of the same letter, the highest power of this letter in the product is produced by the product alone of the first term of the multiplicand by the first of the multiplier; hence, dividing the first term of the product by the first of one of the factors, the quotient must necessarily be the first term of the other factor; therefore, when the dividend and the divisor are arranged according to the powers of any letter, the result of the division of the first term of the dividend by the first term of the divisor is the first term of the quotient.

Let, for example, $A = a^4 + 2a^2b^2 + b^4$ be the dividend, and $B = a^2 + b^2$ the divisor, $\frac{a^4}{a^2} = a^2$ is the first term of the quotient Q.

Now $Q \cdot B$, or $(a^2 + \&c. \dots) B$ ought to give A for product; therefore it must be

$$(a^2 + \dots) B = A;$$

that is, $a^2 \cdot B$ plus the product of B by the remaining terms is equal to A.

In the proposed example,

$$a^2 \cdot B = a^2(a^2 + b^2) = a^4 + a^2b^2;$$

now if from A we subtract $a^4 + a^2b^2$, the remainder will be the product of B by all the terms of the quotient, with the exception of the first; that is to say, the remainder is a product like the given polynomial A, having B for one of its factors, and all the terms of Q except the first for the other factor. Now operating on the remainder as on the given A, we will obtain a second term of the quotient, and also a second remainder, to be divided again, and so on.

§ 47. From the preceding remarks, we easily
Rule and examples. infer the following rule :

To divide a polynomial A by another B, first arrange them according to the powers of the same letter; divide then the first term of A by the first of B, and multiplying B by this quotient, subtract the product from A. Taking then the

remainder for dividend, repeat the same operation until all the terms of the quotient are obtained.

EXAMPLES :—Divide

$$\begin{aligned} (1.) \quad & \begin{cases} A = 2a^4 + 5a^3b + 8a^2b^2 + 7ab^3 + 2b^4 \\ \text{By} \\ B = a^3 + ab + 2b^2. \end{cases} \\ (2.) \quad & \begin{cases} A = 2m^5n + 3m^4n^2 - 8m^3n^3 + 3m^2n^4 - 5n^5 \\ \text{By} \\ B = 2m^2 - mn. \end{cases} \end{aligned}$$

Answers :

$$(1.) \quad Q = 2a^2 + 3ab + b^2.$$

The operation is performed as follows :

(B) Divisor.	(A) Dividend.	Quotient.
$a^3 + ab + 2b^2$	$ 2a^4 + 5a^3b + 8a^2b^2 + 7ab^3 + 2b^4$	$2a^2 + 3ab + b^2$
1st prod. or $2a^2 \cdot B$	$2a^4 + 2a^3b + 6a^2b^2$	
1st rem. or $A - 2a^2 \cdot B$	$.....3a^2b + 4a^2b^2 + 7ab^3 + 2b^4$	$(= R')$
2d prod. or $3ab \cdot B$	$.....3a^3b + 3a^2b^2 + 6ab^3$	
2d rem. or $R' - 3ab \cdot B$	$.....a^2b^2 + ab^3 + 2b^4$	$(= R'')$
3d prod. or $b^2 \cdot B$	$.....a^2b^2 + ab^3 + 2b^4$	
3d rem. or $R'' - b^2 \cdot B$	$.....0$	$(= R''')$

The first term, $2a^4$ of A , divided by the first term a^3 of B , gives $2a^2$ for first term of the quotient ; this multiplied by B , gives the first product, and product subtracted from A , (which is done by changing the signs to the terms of the same product, and adding it to A), gives the first remainder R' . The first term of R' , divided by the first term of B , gives $3ab$, the second term of the quotient, and this term multiplied by B , gives the second product, which subtracted from R' gives the next remainder R'' . Again, the first term of R'' divided by the first term of B , gives b^2 , the third term of the quotient, and $b^2 \cdot B$ subtracted from R'' gives the last remainder $R''' = 0$; hence, the required quotient is

$$Q = 2a^2 + 3ab + b^2.$$

The correctness of the operation can be verified by multi-

plying the quotient by the divisor, which must give the dividend for product. And *vice versa*, the products obtained in the preceding examples of multiplication, divided by the multiplicand, must give, if correct, the multiplier for quotient; and divided by the multiplier, ought to give for quotient the multiplicand; so that each of the preceding examples of multiplication (44) offers two more examples of division for practice, and each of the present examples of division offers another example of multiplication, and one more of division, since the dividend, divided by the obtained quotient, ought to give for result the divisor. So that, in the few examples we have given, and will still give, the industrious learner may find copious matter for practical exercise.

In the first example the dividend is a perfect product, but frequently this is not the case, and instead of having zero for the last remainder, there remain one or more terms which cannot be exactly divided by the divisor; in this case we write for the last term of the quotient, a fraction having the last remainder for numerator and the divisor for denominator. Thus from the second example we will have

$$(2.) \quad Q = m^3n + 2m^2n^2 - 3mn^3 - \frac{6n^5}{2m^2 - mn}.$$

Let us write the whole operation, as in the former case, except the signs, which we will change in order to have them ready for subtraction.

$$\begin{array}{r} 2m^2 - mn \mid 2m^5n + 3m^4n^2 - 8m^3n^3 + 3m^2n^4 - 6n^5 \mid m^3n \\ + 2m^2n^2 - 3mn^3 - \frac{6n^5}{2m^2 - mn} \end{array}$$

1st prod. (Neg.). $-2m^5n + m^4n^2$

$$R' \dots\dots\dots 4m^4n^2 - 8m^3n^3 + \&c.$$

2d prod. (Neg.)..... $-4m^4n^2 + 2m^3n^3$

$$R'' \dots\dots\dots -6m^3n^3 + 3m^2n^4 \&c.$$

3d prod. (Neg.)..... $+6m^3n^3 - 3m^2n^4$

$$R''' \dots\dots\dots -6n^5$$

Other examples. § 48. Divide

$$(3.) \quad \begin{cases} A = b^6 - b^4 + b^3 - b^2 + 2b - 1 \\ \text{by} \\ B = b^3 + b - 1. \end{cases}$$

Answer: $Q = b^4 - b^3 + b^2 - b + 1.$

This example also, presenting some peculiarity in the process of the operation, will be fully developed.

$$\begin{array}{r}
 b^3 + b - 1 \overline{) b^6 - b^4 + b^3 - b^2 + 2b - 1} \quad (b^4 - b^3 + b^2 - b + 1 \\
 \underline{-b^6 - b^3 + b^4} \\
 -b^3 + b^2 - b^2 \\
 + b^3 + b^4 - b^3 \\
 b^4 - b^3 + 2b \\
 \underline{-b^4 - b^3 + b^3} \\
 -b^3 + 2b - 1 \\
 \underline{+ b^3 + b^3 - b} \\
 b^3 + b - 1 \\
 \underline{-b^3 - b + 1} \\
 0
 \end{array}$$

Divide

$$\begin{aligned}
 (4.) \quad & \begin{cases} A = 7a^5b + 22a^4b^2 - 10a^3b^3 + 8a^2b^4 - 30ab^5 \\ \text{by} \\ B = 7a^2b + ab^2 + b^3. \end{cases} \\
 (5.) \quad & \begin{cases} A = 2a^2bm - 2a^2bn + 2a^3cm - 2a^3cn + 3a^4dm - 3a^4dn \\ \text{by} \\ B = 2a^2b + 2a^2c + 3a^4d. \end{cases} \\
 (6.) \quad & \begin{cases} A = a^6 - b^6 \\ \text{by} \\ B = a - b. \end{cases} \\
 (7.) \quad & \begin{cases} A = m^4 - n^4 \\ \text{by} \\ B = m^3 + m^2n + mn^2 + n^3. \end{cases}
 \end{aligned}$$

Answers:

$$(4.) \quad Q = a^3 + 3a^2b - 2ab^2 + \frac{7(a^2b^4 - 4ab^5)}{7a^2b + ab^2 + b^3}.$$

$$(5.) \quad Q = m - n.$$

$$(6.) \quad B = a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5.$$

$$(7.) \quad Q = m - n.$$

ARTICLE III.

Multiplication and Division of Fractions.

§ 49. MULTIPLICATION.—A numerical fraction, for instance $\frac{7}{8}$, designates a number eight times less than 7, because 7 represents seven units, and $\frac{7}{8}$ seven eighth parts of unity; and in general any numerical fraction represents a number less than the numerator, as many times as there are units in the denominator.

Therefore, if, without changing the numerator, the denominator becomes twice, or three, or any number of times greater, the value of the fraction becomes smaller, twice or three times, &c. So, for instance, if we wish to have a fraction half as large as $\frac{7}{8}$, it is enough to take $\frac{7}{16}$, that is $\frac{7}{16}$.

Hence we see the reason of the rules followed in the multiplication of numerical fractions; because when a fraction, for example $\frac{5}{7}$, is multiplied by a whole number, suppose 3, then it is plain that multiplying the numerator alone of the fraction by 3, the product is obtained. We have, namely $1\frac{5}{7}$, which is $\frac{5}{7}$ taken 3 times. But if instead of multiplying $\frac{5}{7}$ by 3, we would have to multiply it by $\frac{3}{5}$; then since 3 is five times greater than $\frac{3}{5}$, the fraction $1\frac{5}{7}$ is also five times greater than the product of $\frac{5}{7}$, by $\frac{3}{5}$; therefore, to reduce $1\frac{5}{7}$ to its just value, we must multiply the denominator 7 by 5, and so we have

$$\frac{5}{7} \cdot \frac{3}{5} = \frac{15}{35} = \frac{5 \cdot 3}{7 \cdot 5};$$

precisely as the rule prescribes, namely, the product of a numerical fraction by another is obtained by multiplying the numerators, and dividing their product by the product of the denominators.

Now the same rule is applicable to algebraical fractions as we will presently see.

Another case is to be mentioned concerning numerical fractions. We have supposed a fraction multiplied by a whole number, and then a fraction multiplied by another fraction; it remains to see how a whole number is multiplied by a fraction; for instance, 6 by $\frac{3}{7}$. The product of 6 by 3, is 18, but 3 is seven times greater than $\frac{3}{7}$; therefore the product of 6 by $\frac{3}{7}$ is the seventh part of 18, that is to say $\frac{18}{7}$; hence

$$6 \cdot \frac{3}{7} = \frac{3}{7} \cdot 6 = \frac{18}{7}.$$

The number 6, and generally any whole number, can be represented also by $\frac{6}{1}$, and giving this fractional form to the whole number, the same rule given for the multiplication of two fractions becomes applicable also to the case of one of the factors being a whole number.

Product of algebraical fractions.

§ 50. Let now the factors $\frac{a}{b}$, $\frac{c}{d}$ be given to be multiplied. The product must be such as to have a numerical value equal to the product of the numerical

First case.

values of the factors; for example, if $\frac{a}{b}$ has $\frac{2}{3}$ for numerical value, and $\frac{c}{d}$ has $\frac{5}{6}$, the product p or $\frac{a}{b} \times \frac{c}{d}$ must have $\frac{2}{3} \times \frac{5}{6} = \frac{10}{18}$ for the corresponding numerical value. Therefore when the terms a , b and c , d have each a whole number for numerical value, the product of the fractions $\frac{a}{b}$, $\frac{c}{d}$ must be obtained in the same manner as that of numerical fractions; that is,

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

Second case. But the terms a, b, c, d may have a fractional numerical value, for instance, $\frac{1}{2}, \frac{2}{5}, \frac{2}{5}, \frac{4}{7}$; in this case we would have

$$\frac{a}{b} = \frac{1}{2} : \frac{3}{5}$$

$$\frac{c}{d} = \frac{2}{5} : \frac{4}{7}$$

Now it is known, and we will soon see it again, that

$$\frac{1}{2} : \frac{3}{5} = \frac{1 \cdot 5}{2 \cdot 3} = \frac{5}{6}$$

$$\frac{2}{5} : \frac{4}{7} = \frac{2 \cdot 7}{5 \cdot 4} = \frac{14}{20}$$

Therefore,

$$\begin{aligned} \frac{a}{b} \times \frac{c}{d} &= \frac{1 \cdot 5}{2 \cdot 3} \times \frac{2 \cdot 7}{5 \cdot 4} \\ &= \frac{1 \cdot 5 \cdot 2 \cdot 7}{2 \cdot 3 \cdot 5 \cdot 4} \\ &= \frac{1 \cdot 2 \cdot 5 \cdot 7}{2 \cdot 5 \cdot 2 \cdot 4} \end{aligned}$$

Observe now that

$$\frac{1 \cdot 2}{2 \cdot 5} : \frac{3 \cdot 4}{5 \cdot 7} = \frac{1 \cdot 2 \times 5 \cdot 7}{2 \cdot 5 \times 3 \cdot 4};$$

hence, from the last member of the preceding equation, we infer

$$\frac{a}{b} \times \frac{c}{d} = \frac{1 \cdot 2}{2 \cdot 5} : \frac{3 \cdot 4}{5 \cdot 7};$$

but

$$\frac{1 \cdot 2}{2 \cdot 5} = \frac{1}{2} \cdot \frac{2}{5}$$

and

$$\frac{3 \cdot 4}{5 \cdot 7} = \frac{3}{5} \cdot \frac{4}{7};$$

therefore,
$$\frac{a}{b} \times \frac{c}{d} = \frac{\frac{1}{2} \cdot \frac{2}{5}}{\frac{3}{5} \cdot \frac{4}{7}}.$$

And since $\frac{1}{2}$ is the numerical value of a , and $\frac{2}{5}$ the numerical value of c , $\frac{3}{5}$ the numerical value of b , and $\frac{4}{7}$ the numerical value of d , we have again

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \cdot c}{b \cdot d},$$

exactly as in the preceding case. Therefore, whatsoever might be the numerical value of the terms of algebraical fractions, the product is obtained always in the same manner; and the rule of multiplication of numerical fractions is applicable in all cases to algebraic fractions. Namely,

Rule. To obtain the product of a fraction multiplied by another, take the product of the numerators and divide it by the product of the denominators.

§ 51. From this it follows that the multiplication of algebraical fractions is the result of the partial products of simple monomials, or polynomials, according as the terms a, b, c, d , &c., are either monomials, or polynomials.

Let us see some examples.

Examples.

Given factors,

- (1.) $\frac{2ab}{m}, \frac{ab^2}{2n}.$
- (2.) $\frac{3lgd}{qr^2}, \frac{f^2qr}{2l^2d^2}.$
- (3.) $\frac{5r^2t}{6ms}, -\frac{l^2m^2n}{10r^2g}.$
- (4.) $-\frac{ce^2s}{3mo}, \frac{5pq}{tv}.$
- (5.) $-\frac{2b^2d^2l}{3sq}, -\frac{3rfq^2}{2b^2d^2l}.$

$$(6.) \quad -\frac{me^3f^4}{ng^3q^3}, -\frac{3g^4q^3sn}{me^3f}.$$

$$(7.) \quad \frac{2a^3+3c}{4m+l}, \frac{2a+b}{2b+n}.$$

$$(8.) \quad \frac{a+b}{m-n}, -\frac{a-b}{m+n}.$$

$$(9.) \quad -\frac{2a^2b+ab^2+b^3}{m^3+2m^2n+m}, \frac{a^2b+ab^3}{3m+2m^3}$$

$$(10.) \quad -\frac{2sg-4tv^3}{m^3}, -\frac{m(a-b)}{r^4}.$$

Answers :

$$(1.) \quad p = \frac{2ab \times ab^3}{m \times 2n} = \frac{2a^2b^3}{2mn} = \frac{a^2b^3}{mn}.$$

$$(2.) \quad p = \frac{3lgqdf^2r}{2qr^3b^3d^3} = \frac{3gf^3}{2rld}$$

$$(3.) \quad p = -\frac{r^2b^3mn}{12st^2g}.$$

$$(4.) \quad p = -\frac{5pqce^3s}{3tvmo}.$$

$$(5.) \quad p = \frac{drfq^3}{sb}.$$

$$(6.) \quad p = \frac{3ef^3g^3q^3s}{1} = 3ef^3g^3q^3s.$$

$$(7.) \quad p = \frac{4a^3+6ac+2a^2b+3be}{8mb+2b+4mn+ln}.$$

$$(8.) \quad p = -\frac{a^3-l^3}{m^3-n^3}.$$

$$(9.) \quad p = -\frac{2a^4b^3+3a^3b^3+2b^4a^3+ab^5}{2m^5+4m^4n+3m^3+6m^2n+2m^2+3m^3}.$$

$$(10.) \quad p = \frac{2asg-4atv^3-2sgb+4btv^3}{m^2r^4}$$

Remarks on the
change of signs.

We may here observe with regard to the example (8) that

$$-\frac{a^2-b^2}{m^2-n^2} = +\frac{b^2-a^2}{m^2-n^2}, \text{ or } = +\frac{a^2-b^2}{n^2-m^2};$$

because, to change a fraction from positive into negative, or *vice versa*, it is enough to change the sign either of the numerator or of the denominator; therefore the fraction $\frac{a^2-b^2}{m^2-n^2}$ becomes negative by merely changing the sign of the numerator; but to change the sign of any quantity means to take the opposite quantity; hence, since b^2-a^2 is the opposite of a^2-b^2 , $\frac{b^2-a^2}{m^2-n^2} = -\frac{a^2-b^2}{m^2-n^2}$. Let the same be said with regard to the denominator. And so also when a double sign is placed before any quantity, for instance, $-(-b)$ or $-(+b)$, the meaning is that the sign next to the term must be changed, and

$$-(-b) = +b$$

$$-(+b) = -b.$$

On the contrary, if the second sign is positive, the first is to be left unchanged; for instance,

$$+(-a) = -a$$

$$+(+a) = a.$$

Product of several
fractions.

§ 52. In the same manner as the product of $\frac{a}{b}$ by $\frac{c}{d}$ is $\frac{a \cdot c}{b \cdot d}$; so the product of $\frac{a}{b}$ by $\frac{c}{d}$ by $\frac{e}{r}$, is $\frac{a \cdot c \cdot e}{b \cdot d \cdot r}$; because the first two factors give $\frac{a \cdot c}{b \cdot d}$ for product, which product multiplied by $\frac{e}{r}$, gives $\frac{a \cdot c \cdot e}{b \cdot d \cdot r}$. The same should be said when the given fractions are four or five, and generally for the product of any number of fractions,

Rule.

Take the product of all the numerators, and divide it by the product of all the denominators.

Examples. Given factors :

$$(11.) \frac{2a^3}{b^2m}, \frac{4abc}{3lq}, \frac{6m^3}{ag^3}.$$

$$(12.) \frac{2a^2b}{3qs^2}, -\frac{fgs}{cd}, \frac{ld^2f^3}{4a^2b^3g^2}$$

$$(13.) -\frac{3qr}{4b^2m^2}, -\frac{sg^2m^3}{blr}, +\frac{gf^3}{dg^2}.$$

$$(14.) \frac{a^2-b}{m}, \frac{3a^2}{q^2-r^2}, \frac{r^2}{ts}.$$

$$(15.) \frac{e^2d}{lm}, \frac{rst}{gf^2}, \frac{t^2d}{m^2}, \frac{r^2es}{t^2f^2}, \frac{td^2}{gv^2}.$$

Answers :

$$(11.) p = \frac{16a^3cm}{blqg^3}.$$

$$(12.) p = -\frac{f^2ld}{6qs^2cabg}.$$

$$(13.) p = \frac{3q^2s^2gf}{4b^2mld}.$$

$$(14.) p = \frac{3a^2r^2-3a^2r^2b}{mstq^2-mtsr^2}.$$

$$(15.) p = \frac{e^2d^2r^2s^2t^2}{l^2m^2g^2f^2v^2}.$$

§ 52. DIVISION.—When any fraction $\frac{a}{b}$ is given to be divided by another $\frac{c}{d}$, the former, or dividend, is to be considered the product of the divisor $\frac{c}{d}$ by another factor. To find this other factor or quotient is to perform the division as we have already seen in the preceding cases of monomials and polynomials.

Now whatever be the numerical value of the Quotient. terms a, b, c, d , the fraction $\frac{a \cdot d}{b \cdot c}$ multiplied by $\frac{c}{d}$

gives for product $\frac{a}{b}$; therefore, $\frac{a \cdot d}{b \cdot c}$ is the quotient of $\frac{a}{b}$ divided by $\frac{c}{d}$. Hence,

$$\frac{a}{b} : \frac{c}{d} = \frac{a \cdot d}{b \cdot c}.$$

And consequently, to obtain the quotient of a fraction divided by another,

Rule. *Take the product of the numerator of the dividend, by the denominator of the divisor, and divide this product by that of the numerator of the divisor multiplied by the denominator of the dividend.*

Exactly as for simple numerical fractions.

§ 54. Divide

Examples.

- (1.) $\frac{2a^2b}{cd}$ by $\frac{d^2b}{acq}$.
- (2.) $\frac{2r^2s}{g^2t^2}$ by $-\frac{fs^2r}{3gt^2}$.
- (3.) $-\frac{mn^5}{7r^2fs}$ by $\frac{3m^2n^3}{28r^2fg}$.
- (4.) $-\frac{dq^2l}{4srt^2}$ by $-\frac{5q^2t^3}{12s^2gt}$.
- (5.) $\frac{13s^4vz^2}{12m^2n}$ by $\frac{s^4vz^2}{24m^2n^2g}$.
- (6.) $\frac{8g^3lt^2}{5rs^2}$ by $\frac{8gl}{15r^2s^2p}$.
- (7.) $\frac{2a^2+b}{m^2n}$ by $-\frac{3f^2}{m-n}$.
- (8.) $-\frac{6v^2s}{rg+d}$ by $\frac{l+2g}{4s^2v}$.
- (9.) $\frac{2a+a^2b}{m+2n}$ by $\frac{s+gt}{q^2+fr}$.
- (10.) $-\frac{a^2+b^2}{m^2+n^2}$ by $-\frac{m^2-n^2}{a^2-b^2}$.

Answers :

$$(1.) \quad q = \frac{2a^3q}{d^3}.$$

$$(2.) \quad q = -\frac{6r^3}{glf^3}.$$

$$(3.) \quad q = -\frac{4n^3rg}{3mf^3}.$$

$$(4.) \quad q = \frac{3dsg}{5r^2q^2b}.$$

$$(5.) \quad q = \frac{26mng}{1} = 26mng.$$

$$(6.) \quad q = 3g^2trsp.$$

$$(7.) \quad q = -\frac{2a^2m - bm - 2a^2n - bn}{3m^2f^2n}.$$

$$(8.) \quad q = -\frac{24v^4s^3}{rgl + dl + 2rg^3 + 2dg}.$$

$$(9.) \quad q = \frac{2aq^3 + a^3bq^3 + 2afr + a^3bfr}{ms + 2ns + mgt + 2ngt}.$$

$$(10.) \quad q = \frac{a^4 - b^4}{m^2 - n^2}.$$

Equivalent fractions.

§ 55. We will terminate this article with a remark concerning the different forms which can be given to the same fraction.

We observed already, that the fraction $\frac{a}{a}$ is equal to unity, and this is equally true whatsoever be the numerical value of a , either whole or fractional; because, for instance, with $a = \frac{3}{4}$, $\frac{a}{a}$ is $\frac{3}{4} : \frac{3}{4} = \frac{3.4}{3.4} = 1$. Now any fraction $\frac{m}{n}$ multiplied by unity gives, as we have seen (49,) $\frac{m.1}{n}$, that is $\frac{m}{n}$; therefore

$$\begin{aligned} \frac{m}{n} &= \frac{m}{n} \cdot \frac{a}{a} = \frac{m}{n} \cdot \frac{a}{a} \cdot \frac{b}{b} \text{ \&c.} \\ &= \frac{ma}{na} = \frac{mab}{nab} \text{ \&c.} \end{aligned}$$

Therefore the numerator and denominator of a fraction can be multiplied by the same quantity without changing the fraction, that is, its value.

But the value of the fraction remains unchanged also, if we divide the numerator and denominator by the same quantity; because in this case we would have, for instance,

$$\text{from } \frac{\frac{m}{a}}{\frac{n}{a}} \text{ namely } \frac{m}{a} : \frac{n}{a};$$

$$\text{but } \frac{m}{a} : \frac{n}{a} = \frac{ma}{na} = \frac{m}{n};$$

$$\text{hence } \frac{\frac{m}{a}}{\frac{n}{a}} = \frac{\frac{m}{a}}{\frac{n}{a}} = \frac{ab}{n} \&c.$$

Hence generally,

The value of a fraction remains unvaried, whether you multiply or divide the numerator and denominator by the same quantity.

CHAPTER III.

FORMATION OF POWERS AND EXTRACTION OF ROOTS.

Formation of powers, and extraction of roots—opposite operations.

§ 56. MULTIPLICATION is a repeated addition of the same quantity to itself, and division is equivalent to a repeated subtraction of the same quantity from another. In like manner the formation of powers and extraction of roots are operations opposite to one another; the first consisting in the repeated multiplication

of the same quantity by itself, and the second being equivalent to the repeated division of a given quantity by the same divisor.

ARTICLE I.

Formation of Powers and Extraction of Roots of Monomials.

Expression of
monomial powers:
definitions.

§ 57. POWERS.—The product of the quantity a by itself, $a . a$, $a . a . a$, &c., is expressed (4) by a^2 , a^3 , &c.; a is termed the *root*; the exponent 2, 3, &c., the *index* or *degree* of the power; and the whole exponential expression a^2 , a^3 , &c., *power*. When the exponent is equal to 2, then the power is also termed the *square*, and when the exponent is equal to 3 the power is also called the *cube*. When the exponent is equal to 4, 5, &c., then the expressions a^4 , a^5 are simply termed the fourth power of a , the fifth power of a , &c.

Composition and
decomposition of
powers having the
same root.

We had occasion to observe, that since a^3 , for instance, multiplied by a^5 is equal to $a . a . a \times a . a . a . a . a$, or equal to a^8 , that is

$$a^3 . a^5 = a^8 = a^{3+5},$$

so for the same reason the power a^9 , for instance, can be decomposed, as follows :

$$a^9 = a^{5+4} = a^5 . a^4.$$

And again, since $a^5 = a^3 . a^2$, $a^4 = a . a^3$,

$$a^9 = a^5 . a^4 = a^3 . a^3 . a . a^3;$$

but

$$a^9 = a^{3+3+1+2}$$

hence also,

$$a^3 . a^3 . a . a^4 = a^{3+3+1+2}.$$

Rule. Generally, *The product of different powers of the same root is obtained by writing once the root, and giving to it for exponent the sum of all the partial exponents.*

For example,

$$b^3 \cdot b^5 \cdot b^3 = b^{3+5+3} = b^{11},$$

$$m^3 \cdot m^3 \cdot m^4 \cdot m^5 = m^{3+3+4+5} = m^{15}.$$

Composition and decomposition of powers having the same index and different root.

§ 58. Let now, for example, a^5 be given to be multiplied by b^5 , both roots a and b having the same index 5. We will have

$$a^5 \cdot b^5 = (ab)^5.$$

In fact, $a^5 \cdot b^5 = a \cdot a \cdot a \cdot a \cdot a \times b \cdot b \cdot b \cdot b \cdot b$ and $(ab)^5 = ab \cdot ab \cdot ab \cdot ab \cdot ab$; in both cases the same number of the same factors; hence, $a^5 \cdot b^5 = (ab)^5$.

In like manner it is easy to see that if any numbers of such powers are to be multiplied, the rule to be followed is,

Rule. *Take the product of the roots, and give to it for exponent the common index.*

For example,

$$m^3 \cdot n^3 \cdot q^3 \cdot s^3 = (m \cdot n \cdot q \cdot s)^3$$

$$r^5 \cdot t^5 \cdot v^5 \cdot p^5 = (r \cdot t \cdot v \cdot p)^5.$$

Powers of powers.

§ 59. A power, like any quantity a or b , can be taken as the root of another power. In this case we will have, for instance,

$$(a^3)^4.$$

Now

$$\begin{aligned} (a^3)^4 &= a^3 \cdot a^3 \cdot a^3 \cdot a^3 \\ &= a^{3+3+3+3}. \end{aligned}$$

But $3 + 3 + 3 + 3 = 3 \cdot 4$; therefore, $(a^3)^4 = a^{3 \cdot 4}$.

And since the same could be demonstrated for any two indices, so we generally infer,

Rule. *The double power is obtained by raising the same root to the product of the powers.*

For example :

$$(m^3)^5 = m^{15}$$

$$(n^4)^6 = n^{24}.$$

Examples. § 60. From the remarks just made it follows that the monomial $3a^3b$ raised to the fourth power, will give

$$(3a^3b)^4 = 3^4 \cdot (a^3)^4 b^4 = 81a^{12}b^4.$$

And in like manner from the given quantities,

$$(1.) (3bg^2)^3 / (7a^3b^2)^2 / (mn^2)^3 \cdot g^2,$$

$$(4.) (mn)^2 \times (mn)^4 / (3m^2s)^2 \times (3m^2s)^2,$$

$$(6.) (a^2b^2)^3 \times (mn^2)^2 \times (ab)^2,$$

$$(7.) (f^2)^4 \times (f^2)^3,$$

$$(8.) (ab)^2 \times (m^2n)^2 \times (a^2b)^2,$$

$$(9.) (a^2b^2m^4f)^2,$$

$$(10.) (m^2)^5 bc^2d^2,$$

we have the answers :

$$(1.) (3bg^2)^3 = 9b^2g^4.$$

$$(2.) (7a^3b^2)^2 = 343a^6b^4.$$

$$(3.) (mn^2)^2 \cdot g^2 = (mn^2 \cdot g)^2 = m^2n^4g^2.$$

And from the others :

$$(4.) (mn)^7.$$

$$(5.) (3m^2s)^7 = 2187 \cdot m^{14}s^7.$$

$$(6.) (a^2b^2mn^2)^2 = a^4b^4m^2n^4.$$

$$(7.) f^{14}.$$

$$(8.) a^2b^2m^4n^2.$$

$$(9.) a^{12}b^{12}m^{24}f^4.$$

$$(10.) m^{25}b^5c^2d^2.$$

Symbol and equivalent expression. § 61. EXTRACTION OF ROOTS.—When the root of any quantity is to be found, we must consider the same quantity as the power of that root. For instance, suppose that we have to find the root of the third degree of the quantity b ; call r this root, then the given quantity b must be equal to r^3 . The symbol by which the third root of b is designated is

$$\sqrt[3]{b},$$

and it is read, third root of b . That is, the index or degree of of the root is placed within the radical sign $\sqrt{}$, and the quantity, whose root is to be designated, under the same sign. Thus, for example,

$$\sqrt[4]{m},$$

designates the fourth root of m . The number within the radical sign is called the *index* or *degree* of the root, and when this number is equal to 2, then it is omitted, and the radical sign alone signifies the *second* or *square* root. For instance,

$$\sqrt{n}, \sqrt{d}$$

represent the square root of n , and the square root of d . When the index is the number 3, then the root is called also *cubic*; for instance,

$$\sqrt[3]{m}$$

represents the *cubic* root of m . The whole expression is termed *radical expression*, or simply *the radical*.

We have seen (59) that $(a^s)^t = a^{st}$, and applying the same rule to fractional exponents, we will have, for instance,

$$(m^{\frac{3}{4}})^5 = m^{\frac{3}{4} \cdot 5} = m^{\frac{3 \cdot 5}{4}}$$

and

$$(m^{\frac{3}{4}})^4 = m^{\frac{3}{4} \cdot 4} = m^3;$$

therefore $\sqrt[3]{b}$ can be represented also by

$$b^{\frac{1}{3}}.$$

Because we remarked that if r represents the third root of b , we must have

$$r^3 = b,$$

but

$$(b^{\frac{1}{3}})^3 = b^1 = b;$$

therefore $b^{\frac{1}{3}}$ as well as r represents the *cube* root of b ; and so we may write,

$$\sqrt[3]{b} = b^{\frac{1}{3}};$$

for both expressions represent the same thing; and since the same observation is applicable to any other case, so generally the expressions

$$\sqrt[m]{a}, a^{\frac{1}{m}}$$

$$\sqrt[m]{a^n}, a^{\frac{n}{m}}$$

are to be considered as identical or equivalent, whatever be the whole number represented by m and n .

The index of the root and the exponent of the power can be multiplied by the same number without changing the value of the radical, and *vice versa*.

§ 62. From this it follows that the radical, for instance,

$$\sqrt[3]{a^5}$$

can be transformed into

$$\sqrt[3 \cdot 2]{a^{5 \cdot 2}} = \sqrt[6]{a^{10}}$$

without changing its value because

$$\sqrt[3]{a^5} = a^{\frac{5}{3}} = a^{\frac{5 \cdot 2}{3 \cdot 2}} = \sqrt[6]{a^{10}},$$

and *vice versa*, if the index of the root and the exponent of the power, or quantity under the radical sign, are both divisible by the same number, they may be divided without changing the value of the radical, for instance,

$$\sqrt[12]{a^9} = \sqrt[4]{a^3},$$

the numbers 12 and 9 being accurately divisible by 3; in fact,

$$\sqrt[12]{a^9} = a^{\frac{9}{12}} = a^{\frac{3}{4} \cdot \frac{3}{3}} = a^{\frac{3}{4}} = \sqrt[4]{a^3}.$$

Examples.

In like manner,

$$\sqrt[11]{a^7 b^{14}} = \sqrt[7]{a b^2},$$

because $a^7 b^{14} = (a b^2)^7$, and therefore,

$$\sqrt[11]{a^7 b^{14}} = \sqrt[11]{(a b^2)^7} = \sqrt[7]{(a b^2)^{\frac{11}{7}}} = \sqrt[7]{a b^2}.$$

So also the radicals

$$\sqrt[6]{a^3 b^6}, \sqrt[11]{a^{17} m^{24}}, \sqrt[24]{a^6 b^2 c^6}$$

can be reduced as follows:

$$\sqrt[2]{a b}, \sqrt[3]{a m^3}, \sqrt[3]{a^2 b c^2}.$$

Radicals reduced to the same index.

On the contrary, by multiplying the index of the root and exponent of the power, we can

easily reduce a given number of radicals to the same index ; for instance, multiplying by 2 the index and exponent of the power of the first of the two following radicals,

$$\sqrt[3]{m^4}, \sqrt[6]{n^5},$$

we will have

$$\sqrt[6]{m^8}, \sqrt[6]{n^{10}},$$

both of the same degree. In like manner, the radicals

$$\sqrt{a^4}, \sqrt[3]{mn^3}, \sqrt[4]{lg}$$

can be changed into

$$\sqrt[12]{a^{48}}, \sqrt[12]{m^4n^8}, \sqrt[12]{l^3g^3},$$

and the radicals

$$\sqrt[3]{a^3}, \sqrt{b^2}, \sqrt[6]{mn}$$

into

$$\sqrt[6]{n^2}, \sqrt[6]{b^3}, \sqrt[6]{mn}.$$

Another inference may be deduced from the same principle, namely,

When the index of the root is equal to the exponent of the power, the value of the radical is the root of the same power.

For example,

$$\sqrt[3]{a^3} = a,$$

$$\sqrt[4]{(ab)^4} = ab.$$

And generally, whatever might be the whole number m ,

$$\sqrt[m]{(rs)^m} = rs;$$

because

$$\sqrt[m]{(rs)^m} = (rs)^{\frac{m}{m}} = rs^1 = rs.$$

So also, since $(a^3b^2a)^5 = a^{15}b^{10}d^5$,

$$\sqrt[5]{a^{15}b^{10}d^5} = a^3b^2d,$$

and

$$\sqrt[m^{21}n^{28}r^{14}]{m^{21}n^{28}r^{14}} = m^3n^4r^2;$$

and generally, *when the index divides exactly the exponent, or exponents of the quantity under the radical sign, the root is obtained by taking this quantity with the exponent, divided by the index.*

Nay, since it sometimes occurs that some of the factors

under the radical sign have exponents divisible by the index of the root, and others not; then those of the terms whose exponents are divisible may be brought out of the radical sign, as in the following example :

$$\sqrt[3]{a^3 b^3 m^3 n} = ab^3 \sqrt[3]{m^3 n}.$$

In fact, $\sqrt[3]{a^3 b^3 m^3 n} = \sqrt[3]{a^3 b^3} \cdot \sqrt[3]{m^3 n},$

and $\sqrt[3]{a^3 b^3} = \sqrt[3]{(ab^3)^3} = ab^3.$

In this and in similar cases, therefore, *those terms whose exponents are divisible by the index of the root may be taken out of the radical sign, after dividing the exponents by the index*; thus we will have

$$\sqrt[4]{m^{15} n^{20} p^3 q^4} = m^3 n^5 \sqrt[4]{p^3 q^4},$$

$$\begin{aligned} \sqrt[4]{a^{12} b^{12} r^3 g^8} &= a^3 b^3 \sqrt[4]{(rg^3)^3} \\ &= a^3 b^3 \sqrt[4]{rg^3}. \end{aligned}$$

Product of different roots.

§ 63. Let us now see how two or more radicals can be multiplied together.

1st case.

The first and simplest case is that of two radicals, having the same index.

For example, $\sqrt[3]{a}, \sqrt[3]{b};$

then we will have $\sqrt[3]{a} \times \sqrt[3]{b} = \sqrt[3]{ab}.$

Because, let us call r and r' the root of a and b , it will be

$$\sqrt[3]{a} \times \sqrt[3]{b} = r \cdot r';$$

but since $r = \sqrt[3]{a}, r' = \sqrt[3]{b}$, we must have also,

$$r^3 = a, r'^3 = b;$$

and consequently, $r^3 \cdot r'^3 = ab,$

or $(r r')^3 = ab;$

hence, $\sqrt[3]{ab} = r r',$

because $r r'$ is the quantity which, raised to the third power, gives ab .

Therefore, the value of the product of $\sqrt[p]{a} \times \sqrt[p]{b}$ is the same as that of $\sqrt[p]{ab}$; hence,

$$\sqrt[p]{a} \times \sqrt[p]{b} = \sqrt[p]{ab}.$$

In like manner, $\sqrt[p]{a} \cdot \sqrt[p]{b} \cdot \sqrt[p]{c} = \sqrt[p]{abc}$;

because, $\sqrt[p]{a} \cdot \sqrt[p]{b} = \sqrt[p]{ab}$;

hence, $\sqrt[p]{a} \cdot \sqrt[p]{b} \cdot \sqrt[p]{c} = \sqrt[p]{ab} \cdot \sqrt[p]{c}$.

And again, $\sqrt[p]{ab} \cdot \sqrt[p]{c} = \sqrt[p]{abc}$.

And generally, whatsoever be the whole number m , and the number of radical factors, we will have

$$\sqrt[p]{a} \cdot \sqrt[p]{b} \cdot \sqrt[p]{c} \dots = \sqrt[p]{a \cdot b \cdot c \dots}.$$

General inference. That is, the product of any number of roots of the same index m , and of different quantities, is equal to the m^{th} root of the product of the different quantities.

2d case.

When the radical factors have different indices, then they may be reduced to the same index as we have seen in the preceding number; and after reduction the product will be obtained as in the first case.

For example, the factors,

$$\sqrt[p]{a}, \sqrt[p]{b^3}, \sqrt[p]{m},$$

reduced to the same index, become

$$\sqrt[p]{a^3}, \sqrt[p]{b^9}, \sqrt[p]{m},$$

and since $\sqrt[p]{a^3} \cdot \sqrt[p]{b^9} \cdot \sqrt[p]{m} = \sqrt[p]{a^3 b^9 m}$,

so $\sqrt[p]{a} \cdot \sqrt[p]{b^3} \cdot \sqrt[p]{m} = \sqrt[p]{a^3 b^9 m}$.

Examples.

In like manner from the factors,

$$\sqrt[m^3]{b}, \sqrt[p]{bm^3n}, \sqrt[p]{m^7n^5b^4},$$

we have $\sqrt[m^3]{b} \times \sqrt[p]{bm^3n} \times \sqrt[p]{m^7n^5b^4} = \sqrt[p]{m^{17}b^{22}n^{19}}$.

And from $\sqrt{a^3b}, \sqrt{a^5b^3}, \sqrt{a^7b^4},$

we have the product $\sqrt{a^{47}b^{50}}.$

And from

$$\sqrt[3]{ql^2n^2}, \sqrt{s^2m^4}, \sqrt[3]{n^2l^2},$$

the product

$$nm^2 \sqrt[3]{ql^2s^2l^2}.$$

Powers of roots.

§ 64. A root, like any other quantity, can be raised to any power; for instance, $\sqrt[3]{ab}$ may be raised to the fourth power, which is expressed by writing

$$[\sqrt[3]{ab}]^4$$

Now

$$\sqrt[3]{ab} = (ab)^{\frac{1}{3}},$$

and consequently, $[\sqrt[3]{ab}]^4 = [(ab)^{\frac{1}{3}}]^4 = (ab)^{\frac{4}{3}}$;

but again,

$$(ab)^{\frac{4}{3}} = \sqrt[3]{(ab)^4};$$

therefore,

$$[\sqrt[3]{ab}]^4 = \sqrt[3]{(ab)^4}.$$

That is, *the power of radical is obtained by raising to the given power, the quantity under the radical sign.*

Rule.

Examples.

$$[\sqrt{mn^2}]^2 = n\sqrt{m}$$

$$[\sqrt[3]{l^2l^2f}]^2 = \sqrt[3]{l^4l^2f^2}$$

$$[\sqrt[5]{m^2n^2b}]^3 = \sqrt[5]{m^6n^6b^3}.$$

Roots of roots.

§ 65. The root of a root can be transformed into a simple radical expression. For example,

$$\sqrt[3]{\sqrt{m^5}} = \sqrt[3]{m^{\frac{5}{2}}} = \sqrt[6]{m^5},$$

and since

$$\frac{5}{2} : 3 = \frac{5}{2 \cdot 3} = \frac{5}{6},$$

and

$$\sqrt[3]{m^{\frac{5}{2}}} = m^{\frac{5}{2} : 3} = m^{\frac{5}{6}},$$

so the compound radical

$$\sqrt[3]{\sqrt{m^5}} = m^{\frac{5}{6}} = \sqrt[6]{m^5}.$$

Now the same reasoning may be applied to any other case,

and supposing v and n to represent any two whole numbers, we will have

$$\sqrt[n]{\sqrt[v]{fg}} = \sqrt[vn]{fg}.$$

Rule. Namely, to obtain the root of another root of a given quantity, take the simple root of the same quantity, having for index the product of the partial indices.

Examples. For instance,

$$\sqrt[5]{\sqrt[3]{a^6b^5}} = \sqrt[15]{a^6b^5}.$$

$$\sqrt[6]{\sqrt{a^2b^4f}} = \sqrt[12]{a^2b^4f} = \sqrt[4]{ab^2} \sqrt[3]{f} = \sqrt[4]{b} \cdot \sqrt[3]{a} \sqrt[3]{f}.$$

$$\sqrt[7]{\sqrt[4]{f^3b^5}} = \sqrt[28]{f^3b^5} = \sqrt[7]{b} \cdot \sqrt[4]{f^3}.$$

For other examples concerning the preceding articles, see at the end of the chapter.

§ 66. The root of a quantity supposes the quantity itself to be a power. (61.) Hence, if the quantity given cannot be the power corresponding to the degree of the root, then the root is called impossible, or *imaginary*.

For example, $\sqrt{-m}$

is an imaginary root, because in this case $-m$ is supposed to be the square of another quantity. But the square of any quantity, either positive or negative, is essentially positive; therefore, $-m$ cannot be the square of any quantity, and $\sqrt{-m}$ is an imaginary root. Let the same be said of $\sqrt[4]{-m}$, $\sqrt[6]{-m}$, &c.; because the fourth power, the sixth power, &c. of any quantity is necessarily positive, and consequently when the quantity under the radical sign is negative, and the index of the root is an even number, the radical expression is an imaginary one.

General formula. Now, suppose $2i$ to represent any even number, the expression

$$\sqrt[n]{-m}$$

will exhibit the general formula of these imaginary radicals.

Transformed. And since $\sqrt[n]{-m} = \sqrt[n]{m} \cdot \sqrt[n]{-1}$, managing this expression like a real radical, we will have

$$\sqrt[n]{m} \cdot \sqrt[n]{-1} = \sqrt[n]{m} \cdot \sqrt[n]{-1},$$

that is, $\sqrt[n]{-m} = \sqrt[n]{m} \cdot \sqrt[n]{-1}$,

and changing i into 1,

$$\sqrt{-m} = \sqrt{m} \cdot \sqrt{-1}.$$

Products and powers of imaginary radicals.

§ 67. In equal manner we have

$$\sqrt{-n} = \sqrt{n} \cdot \sqrt{-1},$$

and consequently,

$$\sqrt{-m} \cdot \sqrt{-n} = \sqrt{m} \cdot \sqrt{n} \cdot [\sqrt{-1}]^2;$$

but $[\sqrt{-1}]^2 = \sqrt{(-1)^2} = (-1)^{\frac{2}{2}} = -1$,
therefore,

$$(f) \quad \sqrt{-m} \cdot \sqrt{-n} = -\sqrt{m} \cdot \sqrt{n} = -\sqrt{mn};$$

and likewise, since $[\sqrt{-m}]^2 = \sqrt{-m} \cdot \sqrt{-m}$,

we have also, by substituting $\sqrt{-m}$, instead of

$$(p) \quad [\sqrt{-m}]^2 = -\sqrt{m \cdot m} = -\sqrt{m^2} = -m$$

But from (f) we have also,

$$\sqrt{-m} \cdot \sqrt{-n} \times \sqrt{-r} = -\sqrt{mn} \times \sqrt{-r},$$

and since $\sqrt{-r} = \sqrt{r} \cdot \sqrt{-1}$, and $\sqrt{mn} \cdot \sqrt{r} = \sqrt{mnr}$, so

$$(f) \quad \sqrt{-m} \cdot \sqrt{-n} \cdot \sqrt{-r} = -\sqrt{mnr} \cdot \sqrt{-1},$$

and changing in (f) $\sqrt{-n}$, and $\sqrt{-r}$ into $\sqrt{-m}$, we have

$$(p) \quad [\sqrt{-m}]^3 = -\sqrt{m^3} \cdot \sqrt{-1}.$$

In equal manner from (f') we infer

$$\begin{aligned}\sqrt{-m} \cdot \sqrt{-n} \cdot \sqrt{-r} \times \sqrt{-s} &= -\sqrt{mnr} \cdot \sqrt{-1} \times \sqrt{-s} \\ &= -\sqrt{mnr} \sqrt{s} \cdot \sqrt{-1} \cdot \sqrt{-1} \\ &= -\sqrt{mnrs} \cdot -1;\end{aligned}$$

therefore,

$$(f'') \quad \sqrt{-m} \cdot \sqrt{-n} \cdot \sqrt{-s} = +\sqrt{mnrs},$$

and changing $\sqrt{-n}$, $\sqrt{-r}$, $\sqrt{-s}$ into $\sqrt{-m}$, we deduce from (f'')

$$(p'') \quad [\sqrt{-m}]^4 = \sqrt{m^4} = m^2.$$

Inference. Two imaginary radicals therefore give a negative, but real product, as (f) and (p) , three give a negative and imaginary product, as (f') and (p') , four give a positive and real product, (f'') and (p'') , five would give again another positive, but imaginary product, six a real one but negative, and so on.

Application. To see some few applications of the inference just deduced, let us take the products of the following factors :

- (1.) $\sqrt{-m}, \sqrt{mn}.$
- (2.) $\sqrt{-a^2b}, \sqrt{-b}, \sqrt{-abc}.$
- (3.) $\sqrt{-m^2n}, \sqrt{-fg}, \sqrt{-fn}, \sqrt{-g}.$

Answers :

- (1.) $-m\sqrt{n}.$
- (2.) $-ab\sqrt{abc} \cdot \sqrt{-1}.$
- (3.) $mnfg.$

ARTICLE II.

Formation of Powers and Extraction of Roots of Polynomials.

Remarks. § 68. THE principles laid down in the article just closed, together with their inferences, are applicable to polynomial and fractional quantities as well as to simple monomials. In the present article, therefore, and in the following, we will merely see some applications of the same principles, and not of all, but only of some of them.

Examples. § 69. POWERS.—The square of the binomial $a + b$ is expressed like that of a simple monomial,

by $(a + b)^2 = (a + b)(a + b),$

and since $(a + b)(a + b) = a^2 + 2ab + b^2,$

thus $(a + b)^2 = a^2 + 2ab + b^2.$

That is, *the square of a binomial, is equal to the square of the first term plus the double product of the first and second terms, plus the square of the second term.*

The cube of the same binomial $(a + b)$ or $(a + b)$ is equal to

$$(a + b)^3 (a + b),$$

consequently equal to

$$(a^2 + 2ab + b^2)(a + b).$$

Now effecting the product, we have

$$a^3 + 3a^2b + 3ab^2 + b^3,$$

therefore, $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$

That is, *the cube of a binomial is given by the cube of the first term, plus thrice the square of the first term by the second*

term plus thrice the first term by the square of the second plus the cube of the second term.

Operating in similar manner, the student may easily find

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5, \text{ \&c.}$$

Find also the square and third power of the trinomial

$$a + b + c.$$

Answers :

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$

$$(a + b + c)^3 = a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3ab^2 + 3ac^2 + 3b^2c + 3bc^2 + 6abc.$$

And again the square of the trinomials

$$2a^2 + ab + 2b^2$$

$$27m - 9m^2 - 3m.$$

Answers :

$$(2a^2 + ab + 2b^2)^2 = 4a^4 + 4a^3b + 9a^2b^2 + 4ab^3 + 4b^4$$

$$(27m - 9m^2 - 3m)^2 = (3m)^2 - 2(3m)^3 - (3m)^4 + 2(3m)^5 + (3m)^6.$$

§ 70. ROOTS.—The rule for the extraction of roots to be observed is inferred from the manner in which the power is formed. Because the object of this operation is to find the root by which the given quantity or power is reduced.

Let us see an example.

The trinomial $R = a^2 - ab + b^2$

is the square root of the polynomial

$$P = a^4 - 2a^2b + 3a^2b^2 - 2ab^3 + b^4.$$

Let us compare the terms of the root R with those of the corresponding power P.

The first term of P is the square of the first term of R ;

therefore, the square root of the first term of P gives the first term of R, namely, a^2 ; subtract now $(a^2)^2$ or a^4 from P, the remainder will be

$$r = -2a^2b + 3a^2b^2 - 2ab^3 + b^4.$$

Now $-2a^2b$, the first term of the remainder, is equal to $-2a^2(ab)$, that is, the double product of the first and second term of R, which, consequently divided by $2a^2$, the double of the first term a^2 , must give for quotient the second $-ab$; hence, dividing the first term of the remainder r by the double of the first term already obtained, we obtain the second term of the root R. If now, $2a^2 - ab$, the double of the first term of the root, and the second be both multiplied by the second $-ab$, which gives $-2a^2b + a^2b^2$, and this product be subtracted from the first remainder r , we have a second remainder

$$r'' = 2a^2b^2 - 2ab^3 + b^4.$$

The first term of this r'' is the product of the double of the first term a^2 of R, by the third term of the same R. Therefore, dividing again the first term of the remainder by $2a^2$, the double of the first of the root, we will obtain the third term of the same root namely, b^2 . Multiplying now the double of the two first terms obtained namely, $2(a - ab)$ plus the third term b^2 , by b^2 , we will have $2a^2b^2 - 2ab^3 + b^4$, which product, subtracted from (r'') , gives the last remainder

$$r''' = 0.$$

From the process just described, we infer the following rule for the extraction of the square root of polynomials:

Arrange the given polynomial according to the powers of a letter, and write for first term of the root the square root of the first term of the polynomial; subtract then from the same polynomial its first term.

To obtain the second term of the root, divide the first term of the remainder by the double of the first term of the root;

subtract then from the first remainder the double product of the first term of the root multiplied by the second, plus the square of the second ; thus we have a second remainder.

To obtain the third term, divide the first term of the second remainder by the double of the first term of the root, subtract then from the second remainder, the double product of the two first terms of the root by the third, plus the square of the third. In this manner we obtain a third remainder, from which, in the same manner as from the two preceding, we will obtain another term for the root, and then another remainder, and so on.

This rule, although not rigorously demonstrated here, is correct. But it will be better understood by a practical application. Let the given polynomial be

§ 71. $P = 9m^6 + 12m^5n - 2m^4n^2 - 4m^3n^3 + m^2n^4$;
Examples.

call R the root, the first term of which is $\sqrt{9m^6} = 3m^3$, write it in the first place next to P, as follows, and let r' , r'' , r''' represent the first, second, and third remainder.

(P) $9m^6 + 12m^5n - 2m^4n^2 - 4m^3n^3 + m^2n^4$ $(3m^3 + 2m^2n - mn^2$ R
 $- 9m^6$

$$r' = \dots 12m^5n - 2m^4n^2 - 4m^3n^3 + m^2n^4$$

$$\quad - 12m^5n - 4m^4n^2$$

$$r'' = \dots - 6m^4n^2 - 4m^3n^3 + m^2n^4$$

$$\quad + 6m^4n^2 + 4m^3n^3 - m^2n^4$$

$$r''' \dots \dots \dots 0$$

Divide now the first term of r' by $2 \cdot 3m^3$, we will have $2m^2n$ for second terms of R; this multiplied by the double of the first term and by itself, gives $12m^5n + m^4n^2$, and these two terms taken from r' give the remainder r'' . Divide the first term of r'' by the same $2 \cdot 3m^3$, or double of the first term of R, we have $-mn^2$ for third term of the root; this multiplied by the double of the two first terms and by itself, gives $-6m^4n^2 - 4m^3n^3 + m^2n^4$, which taken from r'' , gives the

last remainder $r''' = 0$; and the three terms of R found, are the whole of the root of P . Taking in fact the sign of R , we will obtain for power the polynomial P .

Given powers :

$$(1.) R^2 = 9r^2s^2 + 18r^2s^4 + 15r^2s^6 + 6rs^8 + s^{10}.$$

$$(2.) R^2 = 4m^4 + 4m^2n - 9m^2n + 4m^2r - 6mn^2 + 2mnr + 9n^4 - 6n^2r + r^2.$$

$$(3.) R^2 = 81m^{10} - 54m^2n^2 + 27m^6n^6 - 6m^4n^8 + m^2n^{12}.$$

$$(4.) R^2 = 4a^2s^2 - 12a^2sbq + 9b^2q^2.$$

$$(5.) R^2 = a^4b^2 - 4a^2bc - 6a^2bd + 4c^2 + 12cd + 9d^2.$$

Answers :

$$(1.) R = 3r^2s + 3rs^3 + s^5.$$

$$(2.) R = 2m^2 + mn - 3n^2 + r.$$

$$(3.) R = 9m^3 - 3m^2n^2 + mn^6.$$

$$(4.) R = 2a^2s - 3bq.$$

$$(5.) R = a^2b - 2c - 3d.$$

Sometimes the given polynomials are not exact powers, or cannot be arranged according to any letter; still the operation is performed always in the same manner.

Extraction of numerical roots. § 72. The rule which is given to extract numerical roots rests upon the same principles on which rests that of polynomials. There is, however, something peculiar to the extraction of numerical roots.

And first observe that the number, for instance, 999, is the greatest of all numbers of three figures, and 100 is the smallest of them. Now

$$(999)^2 = 998001$$

$$(100)^2 = 10000.$$

That is, the number of figures of the square of a number containing only three figures cannot be smaller than five, nor greater than six. The same thing can be proved of a number of four figures, namely, the square of such number cannot

contain less than seven, nor more than eight figures, and with uniform law the same of all other numbers; that is to say, the figures of the square are always twice, or twice minus one as numerous as those of the root. Thus separating, the squares of the preceding numbers into periods of two and two, commencing on the side of units, as follows,

$$\begin{array}{r} 99, 80, 01 \\ 1, 00, 00, \end{array}$$

we have as many periods as there are figures in the roots of the same numbers; so that after the separation of the figures of the given number into periods, we can immediately know what is the number of the figures in the root of that number. Let, for example,

$$N = 14717086596$$

be a number of which the square root is to be found. This number contains six periods, as follows :

$$1, 47, 17, 08, 65, 96.$$

Therefore the root is composed of six figures.

We will not undertake here to detail the manner in which the principles are applied to find out the practical rule of the extraction of square numerical roots; but merely show the rule itself, with the following example:—Take the given number N , divided into periods, and then write the square *root* of the first period met with, next to the given number as for division. The first period in our case is 1, whose square root is likewise 1; subtract the square of 1, that is 1, from the same first period; then bring down the period 47; and divide the first figure 4 by 2, the double of the root found; we will have 2 for quotient, and, for second figure of the root, annex this figure to the dividend.

1,47,17,08,65,96 121314	
2	47
22	44
24	317
241	241
242	7608
2423	7269
2426	33965
24261	24261
24262	970496
242624	970496

We have 22, which, multiplied by the last figure 2, gives 44 to be taken from 47; the remainder is 3, to which annex the next period 17. Take now 24 for divisor, that is the double of 12, the root already obtained; divide by it the remainder 3, together with the first figure of the period 17, that is, divide 31 by 24, the quotient 1 will be the third figure of the root; annex the same figure to 24, and take from 317, 241×1 , namely, the product of 241 by the last figure 1, the remainder is 76; annex to it the following period 08, and take 242 for divisor of the whole number so far as 8; 242 is the double of the root already obtained, the quotient of this division is 3, and 3 is the fourth figure of the root; annex this number to the last divisor 242, and multiply by the same the whole 2423; subtract it from 7608, the remainder is 339, to which annex the next period 65, and perform the division by the double of the root obtained as usual; we will have the next figure of the root, and so likewise the last.

Find the square root of the numbers,
Examples.

- (1.) $N = 104976$.
- (2.) $N = 5499025$.
- (3.) $N = 173056$.
- (4.) $N = 341056$.
- (5.) $N = 1290496$.

- (6.) $N = 1223236.$
- (7.) $N = 857476.$
- (8.) $N = 732736.$
- (9.) $N = 116281.$
- (10.) $N = 4393216.$

Answers:

- (1.) $R = 324.$
- (2.) $R = 2345.$
- (3.) $R = 416.$
- (4.) $R = 584.$
- (5.) $R = 1136.$
- (6.) $R = 1106.$
- (7.) $R = 929.$
- (8.) $R = 856.$
- (9.) $R = 341.$
- (10.) $R = 2096.$

§ 73. If the given number is not a perfect square, then by applying the rule to extract the root from it we will find the square root of the greatest square contained in the given number; and, adding to the same given number decimal ciphers, we can continue the operation, approaching more and more to the root of the given number without ever reaching it; for the numbers that have no root among the whole numbers have none either among fractional ones. The roots of these numbers are consequently called *incommensurable*, or *irrationals*. All this can be rigorously demonstrated, as may be seen in the larger treatise. Here we need not dwell upon this subject.

ARTICLE III.

Formation of Powers and Extraction of Roots of Fractional Quantities.

Formation of powers of fractions the same as that of monomials.

§ 74. POWERS.—It remains now for us to see how the powers of fractions are formed.

This is evidently seen by observing that

$$\left(\frac{a}{b}\right)^2,$$

for example, is equal to

$$\frac{a}{b} \times \frac{a}{b} \times \frac{a}{b}.$$

But the product of fractions is obtained by dividing the product of the numerators by that of the denominators; therefore,

$$\frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a \cdot a \cdot a}{b \cdot b \cdot b} = \frac{a^3}{b^3};$$

hence,

$$\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}.$$

The same may be said of any other power; and representing by n any whole number, we will have

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

Rule. That is, the power of any fraction $\frac{a}{b}$ is obtained by dividing the n^{th} power of the numerator by the n^{th} power of the denominator.

Therefore, the formation of powers of an algebraical fraction is nothing more than the formation of powers of two simple monomials; and if the numerator and denominator of the given fraction are polynomials, the formation of the power of the fraction consists in making separately the formation of the power of the numerator and that of the denominator.

§ 75. EXTRACTION OF ROOTS.—The same is to be said of the extraction of roots.

Suppose, for example, that $\frac{m}{n}$ represents the cube root of $\frac{a}{b}$; namely,

$$\sqrt[3]{\frac{a}{b}} = \frac{m}{n};$$

then

$$\left(\frac{m}{n}\right)^3 = \frac{a}{b},$$

and consequently $a = m^3$, $b = n^3$,

and therefore, $\sqrt[3]{a} = m$, $\sqrt[3]{b} = n$,

and

$$\frac{\sqrt[3]{a}}{\sqrt[3]{b}} = \frac{m}{n};$$

both therefore $\sqrt[3]{\frac{a}{b}}$ and $\frac{\sqrt[3]{a}}{\sqrt[3]{b}}$ are equivalent to $\frac{m}{n}$; hence,

$$\sqrt[3]{\frac{a}{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}}.$$

The same can be proved of any other root; and representing again by n any whole number, we will have likewise,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

Rule. That is, the n^{th} root of a fraction is given by the n^{th} root of the numerator, divided by the n^{th} root of the denominator.

MISCELLANEOUS EXAMPLES FOR PRACTICE.

§ 76. BEFORE commencing the second part, we add here some examples for practice of the rules explained in the preceding articles.

MULTIPLICATION.

Given factors :

- (1.) $A = 7a^3 - 5a^2b + 6ab^2 - 2b^3,$
 $B = 3a^4 - 4a^3b + 16a^2b^2.$
- (2.) $A = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5,$
 $B = a^3 - 3a^2b + 3ab^2 - b^3.$
- (3.) $A = 5a^2b^2c^2 - 6a^4b^3c^2 + 7a^5b^5c^2,$
 $B = 2a^2b^2c^2 + 3a^4b^2c^2 - 6a^7b^4c^2.$
- (4.) $A = x^3 + (a+1)x^2 - (a^2+2a-3)x + (a^3-5a^2+8a-7),$
 $B = x^2 + (a-1)x + (a^2-3a+1).$
- (5.) $A = y^3 + (a+b)y^2 + (a^2-b^2)y + a^2-3ab+3ab^2-b^2,$
 $B = y^2 - (a-b)y + a^2-2ab+b^2.$

Answers :

- (1.) $P = 21a^7 - 43a^6b + 150a^5b^2 - 110a^4b^3 + 104a^3b^4 - 32a^2b^5.$
- (2.) $P = a^8 - 8a^7b + 28a^6b^2 - 56a^5b^3 + 70a^4b^4 - 56a^3b^5 + 28a^2b^6 - 8ab^7 + b^8.$
- (3.) $P = 10a^8b^4c^4 + 3a^7b^5c^7 + 14a^{11}b^3c^2 - 18a^9b^4c^{10} + 21a^{12}b^7c^{11} - 30a^{10}b^7c^5 + 36a^{11}b^8c^8 - 42a^{12}b^9c^3.$
- (4.) $P = x^5 + 2ax^4 + (a^2-5a+3)x^3 + (a^3-8a^2+11a-9)x^2 - (5a^3-21a^2+2ba-10)x + a^5+8a^4+24a^3-36a^2+29a-7.$
- (5.) $P = y^5 + 2by^4 + (a^2-2ab+b^2)y^3 + (a^3-3a^2b+3ab^2-b^3)y^2 + 2(a^2b-3a^2b^2+3ab^3-b^4)y + a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5.$

§ 77. DIVISION.

$$(1.) \frac{72x^4 - 78x^3y - 10x^2y^2 + 17xy^3 + 3y^4}{6x^2 - 4xy - y^2}.$$

$$(2.) \frac{a^4 - 9a^2b^2 - 6abc^2 - c^4}{a^2 - 3ab - c^2}.$$

$$(3.) \frac{3a^5 + 16a^4b - 33a^3b^2 + 14a^2b^3}{a^2 + 7ab}.$$

$$(4.) \frac{33ab + 18a^2 - 12ad - 30b^2 + 42ac + 124bc + 8bd - 16c^2 - 32cd}{6a + 15b - 2c - 4d}.$$

$$(5.) \frac{4a^2 + (6b - 4x)a + 9bx - 15x^2}{2a + 3b - 5x}.$$

Answers :

$$(1.) Q = 12x^2 - 5xy - 3y^2.$$

$$(2.) Q = a^2 + 3ab + c^2.$$

$$(3.) Q = 3a^3 - 5a^2b + 2ab^2.$$

$$(4.) Q = 3a - 2b + 8c.$$

$$(5.) Q = 2a + 3x.$$

FRACTIONS.

§ 78. ADDITION.—Given terms :

$$(1.) \frac{a^2}{(a+b)^2} - \frac{ab}{(a+b)^2} + \frac{b}{a+b}.$$

$$(2.) \frac{3a+2x}{a+x} - \frac{5a-x}{a-x} + \frac{a}{2x}.$$

$$(3.) \frac{a}{b+x} - \frac{c}{x} + \frac{3c}{4x} + 2b.$$

$$(4.) \frac{a}{a+z} + \frac{z}{a-z}.$$

$$(5.) \frac{f+g}{3f-2g} - \frac{5f-2g}{2f-9g}.$$

Answers :

$$(1.) \frac{a^3 + bab^2 + b^3}{(a+b)^3}.$$

$$(2.) \frac{a^3 - 4a^2x - 11ax^2 - 2x^3}{2x(a^2 - x^2)}.$$

$$(3.) \frac{8bx^2 + (8b^2 + 4a - c)x - bc}{4x(x+b)}.$$

$$(4.) \frac{a^3 + x^3}{a^3 - x^3}.$$

$$(5.) \frac{9fg - 13f^2 - 13g^2}{6f^3 - 31fg + 18g^3}.$$

§ 79. MULTIPLICATION AND DIVISION.—Given :

$$(1.) \left[\frac{a+x}{a-x} + \frac{b-x}{b+x} \right] \times \left[\frac{a-x}{a+x} + \frac{b+x}{b-x} \right].$$

$$(2.) \left[\frac{3a+2x}{2a-x} + \frac{4x}{2a} \right] \times \left[\frac{5a-3x}{4a-x} + \frac{7x}{5a} \right].$$

$$(3.) \left(\frac{a}{b} + \frac{c}{d} \right) : \left(\frac{e}{f} + \frac{g}{h} \right).$$

$$(4.) \left(\frac{a}{b} + \frac{c}{d} + \frac{e}{f} \right) : \left(\frac{g}{h} + \frac{i}{k} + \frac{l}{m} \right).$$

$$(5.) \left(\frac{x}{x-a} + \frac{a}{x+a} \right) : \left(\frac{x}{x-a} - \frac{a}{x+a} \right).$$

Answers :

$$(1.) \frac{4(ab+x^2)^2}{(a^2-x^2)(b^2-x^2)}.$$

$$(2.) \frac{150a^4 + 378a^3x + 14a^2x^2 - 136ax^3 + 28x^4}{10a^2(8a^2 - 6ax + x^2)}.$$

$$(3.) \frac{(ad+bc)fh}{(eh+fg)bd}.$$

$$(4.) \frac{(adf + bcf + bde)hkm}{(gkm + him + hkl) bdf}.$$

$$(5.) \frac{x^3 + 2ax - a^3}{x^3 + a^3}.$$

CALCULUS OF RADICALS.

§ 80. Find the equivalent expressions of the following radicals :

$$(1.) \sqrt{\frac{a^3c}{b^3}} + \sqrt{\frac{a^3c^3}{bd^3}} - \sqrt{\frac{a^3cd^3}{be^3}}.$$

$$(2.) 3b^3\sqrt{a^3c} + \frac{2}{c}\sqrt{a^3c^3} - c^4\sqrt{\frac{ac}{b^3}}.$$

$$(3.) \sqrt{\frac{a^3b^3}{cd^3} - \frac{2a^3b^3}{c^3d}}.$$

$$(4.) \sqrt{3a^3c + 6abc + 3b^3c}.$$

$$(5.) \sqrt{4a^3b^3 - 20a^3b^3 + 25ab^3}.$$

$$(6.) \sqrt{\frac{a^3x - 2ax^3 + x^3}{a^3 + 2ax + x^3}}.$$

Answers :

$$(1.) \left(\frac{a^3}{b} + \frac{ac}{d} - \frac{ad}{e}\right)\sqrt{\frac{c}{b}}.$$

$$(2.) \left(3ab^3 + 2a^3 - \frac{c^4}{b}\right)\sqrt{ac}.$$

$$(3.) \frac{ab}{cd}\sqrt{ac - 2bd}.$$

$$(4.) (a + b)\sqrt{3c}.$$

$$(5.) (2a^3b - 5b^3)\sqrt{a}.$$

$$(6.) \frac{a - x}{a + x}\sqrt{x}.$$

§ 81. PRODUCTS AND QUOTIENTS.—Given:

$$(1.) \left[\sqrt{\frac{ad^3}{c^3}} + \sqrt{\frac{a^3}{b}} \right] \times [\sqrt{ac} + \sqrt{b^3}].$$

$$(2.) \left[\sqrt{\frac{ac^3}{a+b}} + \frac{1}{\sqrt{b}} \right] \times \left[\frac{c}{d} \sqrt{(a+b)a} - \sqrt{\frac{b^3}{c^3}} \right].$$

$$(3.) (a + b\sqrt{-1})(a - b\sqrt{-1}).$$

$$(4.) (x - a + b\sqrt{-1})(x - a - b\sqrt{-1}).$$

$$(5.) \sqrt{a} : (b + \sqrt{c}).$$

$$(6.) \sqrt{a} : (\sqrt{b} + \sqrt{c}).$$

$$(7.) (a + b\sqrt{-1}) : (a - b\sqrt{-1}).$$

$$(8.) (m + n\sqrt{-1}) : (p + q\sqrt{-1}).$$

Answers:

$$(1.) \frac{ad}{c} + ab + \left(a + \frac{b^2d}{c^3}\right) \sqrt{\frac{ac}{b}}.$$

$$(2.) \frac{ac^3}{d} - \frac{b^3}{c} + \left(\frac{c}{bd} - \frac{b^3}{a+b}\right) \sqrt{(a+b)ab}.$$

$$(3.) a^2 + b^2.$$

$$(4.) (x - a)^2 + b^2.$$

$$(5.) \frac{b\sqrt{a} - \sqrt{ac}}{b^2 - c}.$$

$$(6.) \frac{\sqrt{ab} - \sqrt{ac}}{b - c}.$$

$$(7.) \frac{a^2 - b^2}{a^2 + b^2} + \frac{2ab}{a^2 + b^2} \sqrt{-1}.$$

$$(8.) \frac{mp + nq}{p^2 + tq^2} + \frac{np - mq}{p^2 + q^2} \sqrt{-1}.$$

PART SECOND.

PRELIMINARY ARTICLES.

Division of the second part. § 82. THE second part contains three chapters, like the first; in the first of which we will treat of the equations of the first degree; in the second, of the equations of the second degree; in the last, of proportions and progressions.

Resolution of the equations and general principles. 83. Most of the questions that are treated of in equations have for object the resolution, either total or partial, of the same equations. To resolve an equation is to find out the value of the *unknown* quantity or quantities which are in it. For example, in the equation

$$2 + 4 + x = 10,$$

the numbers 2, 4, together with x , which is not given, must make a sum equal to 10; but if we take

$$x = 4,$$

then indeed we have $2 + 4 + x = 10$. Now, to find out this value of x , which makes the first number equal to the second, is to resolve the equations; so likewise, in the equations

$$2x + y + 30 = 42$$

$$x + y + 4 = 11$$

every thing is known except x and y ; to find what value is to be given to these unknown quantities in order to fulfil at once the condition that the first member of the first equation be equal to 42, and the first member of the second equation be equal to 11, is to resolve the equations. Hence, we say that to resolve one or more equations is to find out the values of the

unknown quantities; hence, when substituting instead of the unknown quantities such values as fulfil the conditions of the equations, the equations are said to be resolved. Thus, for example, since taking

$$\begin{aligned}x &= 5 \\ y &= 2,\end{aligned}$$

and making the substitution of 5 and 2, instead of x and y , in the preceding equations, we have the first member of the first equation equal to 42, and the first of the second equal to 11; so the same equations are resolved by

$$x = 5, y = 2.$$

Symbols used for known and unknown quantities.

§ 84. The symbols commonly used to represent unknown quantities are the last letters of the alphabet, such as x, y, z ; the other letters are used to represent known quantities.

Axioms to be frequently used in the resolution of equations.

§ 85. 1st. When two or more more quantities are equal to another quantity, they are all equal among themselves; for example,

$$\begin{aligned}m &= b \\ n &= b \\ p &= b;\end{aligned}$$

hence,

$$m = n = p.$$

Let the quantities m and n be equal to one another, we will have

$$(p) \quad m = n,$$

and consequently,

$$2d. \quad m + a = n + a.$$

That is, *if to two equal quantities the same quantities be added, the sums are equal.*

3d. From the same (p) , we evidently have

$$m - a = n - a.$$

That is, *when from equal quantities the same quantity is subtracted, the remainders are equal.*

4th. From (p) again, we have

$$a \times m = a \times n;$$

that is, *two equal quantities, multiplied by the same quantity, give equal products.*

5th. If, finally both members of (p) be divided by a , we will manifestly have

$$\frac{m}{a} = \frac{n}{a};$$

that is, *when two equal quantities are divided by the same quantity, the quotients are equal.*

The same may also be said of raising to powers and the extracting of roots.

§ 86. Let us now see what use may be made of these axioms. Take the equation

$$(t) \quad 2x + b - a = l + \frac{1}{4}s.$$

According to the first axiom, we may add the same quantity to both members, having still the first member of the equation equal to the second; thus, from (t), adding a to both members, we have

$$2x + b - a + a = l + \frac{1}{4}s + a;$$

and consequently,

$$(t') \quad 2x + b = l + \frac{1}{4}s + a.$$

According to the second axiom, the same quantity being subtracted from both members of the equations, the remainders are equal; thus, taking from both members of (t') the quantity $(b + l)$, we will have

$$2x + b - a - b - l = l + \frac{1}{4}s - b - l;$$

and since $b - b$ and $l - l$ are equal to zero, so the last equation is equivalent to

$$(t'') \quad 2x - a - l = \frac{1}{4}s - b.$$

Compare now (t') and then (t'') with (t).

(t') differs from (t) in this, that the same term $-a$, which

is in the first member of (t) , is changed into $+a$ and transposed to the second member of (t') .

(t') differs from (t) in this, that the term $+b$ of (t) is changed into $-b$ in (t') and transposed to the second member: again the term $+l$ of (t) is changed into $-l$ in (t') and transposed from the second to the first member. Therefore, we infer generally,

1st Rule. Any term of an equation may be transposed from one member to the other by changing its sign.

From this we deduce two corollaries:

1st. All the terms of the first member may pass to the second, and all those of the second to the first, provided their signs be all changed; that is,

The signs of all the terms of an equation can be changed without destroying the equality of the members.

For example, the equation

$$3yb - mq + d = l + r - s$$

can be changed into

$$mq - 3yb - d = s - l - r.$$

2d. If the same term or terms are to be found in both members of a given equation, and affected with the same sign, transposing those of the first member to the second, they will be destroyed by the opposite terms. That is,

Equal terms affected with the same sign in both members of the equation, can be eliminated without altering the equality of the members.

So, for example, from

$$2ax + b - c = m + b$$

we infer

$$2ax - c = m.$$

It is plain, moreover, that if all the terms of the second member, for example, are transposed to the first, the second member becomes zero; so from

$$2axb - cd = l + v - q$$

we have

$$2axb - cd - l - v + q = 0.$$

But let us return to the equation (t). According to the third axiom, both members of a given equation can be multiplied by the same quantity, remaining yet equal to one another; hence, from (t) we have

$$4(2x + b - a) = 4(l + \frac{1}{4}s);$$

that is $(t'') \quad 8x + 4b - 4a = 4l + s.$

And comparing (t'') with (t), we see the term $\frac{1}{4}s$ of (t) is cleared of the denominator 4 in (t'') , and all the other terms are multiplied by this denominator. Hence,

2d Rule. Any term of the given equation having a denominator can be cleared of it, by multiplying all the other terms by the same denominator.

From the last axiom we see that both members of an equation can be divided by the same quantity without destroying the equality; therefore, dividing both members of (t) by 2, we have

$$\frac{2x + b - a}{2} = \frac{l + \frac{1}{4}s}{2},$$

from which

$$(t''') \quad x + \frac{1}{2}b - \frac{1}{2}a = \frac{1}{2}l + \frac{1}{2 \cdot 4}s;$$

and comparing (t''') with (t), we can easily remark that the term $2x$ of (t) in (t''') is cleared of the coefficient 2, while all the other terms become divided by the same coefficient. Hence,

3d Rule. Any term of a given equation having a coefficient may be cleared of it by dividing all the other terms by that coefficient.

This rule holds good also in the case of a fractional coefficient; because, considering separately the numerator and denominator of the coefficient, dividing by the former all the other terms according to the second rule, and multiplying by the latter the same terms, we will have the term cleared of the fractional coefficient, while all the others become multiplied

by the same fraction, but with inverted terms. But to multiply any quantity by a fraction taken with inverted terms is to divide that quantity by the original fraction; therefore the rule is general, and the term x of the equation

$$\frac{1}{3}x - b = m + n$$

can be cleared of its coefficient by multiplying all the other terms by $\frac{1}{3}$, namely,

$$x - \frac{1}{3}b = \frac{1}{3}m + \frac{1}{3}n.$$

CHAPTER I.

EQUATIONS OF THE FIRST DEGREE.

What the degree of an equation is. § 87. THE degree of an equation is the same as the highest degree or power of the unknown quantities that are in the same equation; for example, the equation

$$x^3 + y^2 - b = m + q,$$

which contains the unknown quantities x and y , the first raised to the third power, the second squared, is an equation of the third degree. So also the equation

$$x^2 + b^3 + c^4 = m + n,$$

is an equation of the second degree, because the power of the unknown quantity x is 2, and although b is raised to the third, and c to the fourth power, yet since b and c are known quantities, their powers do not determine the degree of the equation. Hence, for practice, see what is the highest exponent of the unknown quantity or quantities in a given equation: the same is the degree of the equation.

Equations of the first degree and their classes.

§ 88. We easily see now that any equation whose unknown quantity or quantities have no

other exponent except unity, is an equation of the first degree. We have said quantity or quantities, because the same equation, as we have already seen, can contain one or more unknown quantities. But the resolution of those equations which contain more than one unknown quantity necessarily differs from that of those which contain only one unknown quantity; hence, we will divide this chapter into two articles, in the first of which we will resolve the equations containing only one unknown quantity—in the second we will resolve the equations containing more unknown quantities.

ARTICLE I.

Resolution of the Equations of the first degree containing only one unknown quantity.

§ 89. THE operation to be performed to obtain the resolution of the equations of the first degree containing a single unknown quantity is nothing more than a bare application of the rules given in the preceding articles. Let us see an example :

The equation $2x - b = q - r$

is certainly resolved, provided the unknown quantity x remains alone and cleared of its coefficient in the first member; because then the other member being entirely composed of known quantities is a known quantity itself, and having besides the same value of x ; therefore, x also becomes a known quantity, and the equation is resolved.

Now, according to the first of the above-mentioned rules, $2x$ may be left alone in the first member by bringing b with changed sign to the second member; and $2x$, according to the second rule, may be cleared of its coefficients by dividing all

the other terms by 2. And, consequently, the unknown quantity x will be given by

$$x = \frac{b + q - r}{2}.$$

Suppose $b = 4$, $q = 6$, $r = 7$, then the given equation becomes $2 \cdot x - 4 = 6 - 7 = (-1)$;

and the value of x ,

$$x = \frac{4 + 6 - 7}{2} = \left(\frac{3}{2}\right);$$

substituting, in fact, in the preceding equation, $\frac{3}{2}$ instead of x , we have

$$2 \cdot \frac{3}{2} - 4 = 6 - 7;$$

but $2 \cdot \frac{3}{2} - 4$ as well as $6 - 7$ is equal to -1 ; therefore, the value $\frac{3}{2}$ found for x fulfils the condition of the equation, and the equation is resolved.

From this example we may pass to resolve easily the following equations. Clearing, namely, the unknown quantity of its coefficient, and leaving it alone, either in the first or second number.

§ 90. Examples :

First case.

- (1.) $\frac{1}{4}x - 4 = 19.$
- (2.) $7 - 2x + 4 = -42.$
- (3.) $13x - 18 + \frac{1}{2} = 4.$
- (4.) $16 - \frac{3}{4}x + 5 = 0.$
- (5.) $a - 6x = m - n.$

Answers :

- (1.) $x = 92.$
- (2.) $x = 26 + \frac{1}{2}.$
- (3.) $x = \frac{43}{26}.$
- (4.) $x = 28.$
- (5.) $x = \frac{a + n - m}{6}.$

Sometimes the unknown quantity is to be found in various terms of the equation, as in the following cases :

$$ax + b - cx = m + n$$

$$ax - m + q = x - c,$$

bring then first all the terms which contain the unknown quantity to one member, and all the others to the other member; in this manner the given equations will be transformed into

$$ax - cx = m + n - b$$

$$ax - x = m - q - c;$$

the unknown quantity then becomes necessarily the common factor of all the terms contained in the same member; and, consequently, this member can be expressed by two factors, one of which is the unknown quantity, and the other the sum of all the quantities originally multiplied by it; so the first member of the last equations are equivalent, the one to $x(a - c)$, the other to $x(a - 1)$. Considering now the factors $(a - c)$ or $(a - 1)$ or any other in similar cases as the coefficient of x , the given equations will be resolved as in the case of a single term containing the unknown quantity. Thus, from the proposed examples, or

$$x(a - c) = m + n - b$$

$$x(a - 1) = m - q - c,$$

we will have

$$x = \frac{m + n - b}{a - c},$$

$$x = \frac{m - q - c}{a - 1}.$$

But let us give some other examples :

$$(1.) \quad 8x - 5 = 13 - 7x.$$

$$(2.) \quad 2x + 7 + \frac{3}{2}x = 6x - 23.$$

$$(3.) \quad \frac{3x}{5} - \frac{7x}{10} + \frac{3x}{4} - \frac{7x}{8} + 15 = 0.$$

$$(4.) \quad \frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 7x - 712 + \frac{x}{5}.$$

$$(5.) \quad \frac{16}{5}x + 132 - x = \frac{19}{3}x + 7345 - \frac{2}{3}x.$$

Answers :

$$(1.) \quad x = 1 + \frac{1}{5}.$$

$$(2.) \quad x = 12.$$

$$(3.) \quad x = 66 + \frac{2}{3}.$$

$$(4.) \quad x = 116 + \frac{148}{367}.$$

$$(5.) \quad x = \frac{108195}{52}.$$

Another case may happen: that, namely, the **Third case.** unknown quantity be in the denominator or denominators of some of the terms of the given equation. But, according to the rule, any term can be cleared of the denominator by multiplying all the other terms by that denominator; and, consequently, any equation of this last description is easily reduced to the form of those of the first, or those of the second case. Thus, for example, the equation

$$\frac{a}{x} - \frac{2b}{3x} + m - q = d,$$

is first transformed into

$$a - \frac{2b}{3} + mx - qx = dx,$$

and then resolved as usually, it gives

$$x = \frac{a - \frac{2}{3}b}{d + q - m}.$$

Other examples :

$$(1.) \quad 16 + \frac{3}{x} + \frac{1}{2} - 12 = 4 - \frac{2}{x} + 2.$$

$$(2.) \quad \frac{10}{x} - 15 = -\frac{8}{x} - 9.$$

$$(3.) \quad \frac{1}{x} - 19 + \frac{18}{x} - 12 + \frac{58}{x} = -15 + \frac{5}{x}.$$

$$(4.) \quad \frac{4}{x} - \frac{6}{5} = \frac{18}{15} - \frac{8}{x}.$$

$$(5.) \quad 18 - \frac{3}{x} + 4 = 6 - \frac{3}{x} + 4 - \frac{60}{x}.$$

Answers :

$$(1.) \quad x = 2.$$

$$(2.) \quad x = 3.$$

$$(3.) \quad x = 4.$$

$$(4.) \quad x = 5.$$

$$(5.) \quad x = 5.$$

Problems. § 91. In the same manner as equations, problems are resolved.

What a problem is. A problem is an enigmatic question, in which something is given or known; some other is unknown and to be found. But when the problem is proposed, those things that are unknown, and those that are known are made to depend on each other by some expressed condition. Now, the same mutual connection expressed by words in the problem can be represented by an algebraical equation, composed of the known and unknown quantities of the problem; so that resolving this equation, the problem also is resolved. The difficulty, therefore, proper to the resolution of problems, consists in finding out the equation expressive of the problem.

Examples. § 92. Find the number which, subtracted
First problem. twice from 90, gives a difference equal to itself.

Let us call x the unknown number, $2x$ is twice the same number. But taking twice x from 90, the remainder is x ; therefore, the equation contained in the problem is

$$90 - 2x = x;$$

from which we have $x = 30$;

taking, in fact, twice 30 from 90, the remainder is 30.

Remark. Let us observe, that to render easier the solution of the problem, it is frequently expedient to point out first what is or are the unknown quantities of the problem, and designate them immediately by appropriate symbols. Then, to modify the same symbols, and combine them with the known terms, in the manner described by the problem.

Problem 2. Let us see another example: 2500 dollars have been given to two persons, but the first received as many times 20 dollars as the second received 5. How many dollars did each receive?

This problem seems to contain two unknown quantities, but in reality there is only one; because, if we know how many times the sum of twenty dollars is to be given to the first, the part belonging to the second is likewise known.

Let us call therefore x this number of times. The part to be given to the first man is then

$$20 \cdot x,$$

and that to be given to the second is

$$5 \cdot x;$$

both together make the sum of 2500; therefore, the equation equivalent to the problem is

$$20x + 5x = 1500,$$

from which $x = 100$.

Now $20 \cdot 100 = 2000$,

and $5 \cdot 100 = 500$;

the sum given to the first man therefore is 2000, and that given to the second is 500 dollars.

Divide the number 237 into two such parts that

Problem 3. the one be four times greater than the other.

Answer :

Calling x the smaller part, we have the equation

$$4x + x = 237,$$

and consequently

$$x = 47 + \frac{2}{5}$$

$$4x = 189 + \frac{8}{5}.$$

Problem 4. The fourth part plus the fifth part of the soldiers forming an army amount to 18000. What is the number of soldiers in the army?

Answer :

The equation expressing the problem is

$$\frac{x}{4} + \frac{x}{5} = 18000,$$

and the required number

$$x = 40000,$$

Problem 5. Find out the number which divided first by 4, and then by 6, gives two such quotients that added together they make 9.

Answer :

The equation is $\frac{x}{4} + \frac{x}{6} = 9.$

And the number $x = 21 + \frac{3}{5}.$

For other problems see at the end of the book.

ARTICLE II.

Resolution of the Equations of the first degree, containing various unknown quantities.

§ 93. LET a given equation contain two unknown quantities, x and y , as, for example, in the following case :

$$(e) \quad ax + by = m;$$

this equation, resolved with regard to x , gives

$$x = \frac{m}{a} - \frac{b}{a}y;$$

and resolved with regard to y , it gives

$$y = \frac{m}{b} - \frac{a}{b}x;$$

therefore, the value of x is given by an expression involving the unknown quantity y , and the value of y is given by a similar expression, containing the unknown quantity x . Therefore neither of them is determined; and unless either x or y , or both of them are connected with other quantities to form another equation, their value remains necessarily undetermined. Suppose then, that, together with the first equation (e), another equation (e') is given, with the same unknown quantities, as follows :

$$(e') \quad lx + gy = n;$$

then the value of x and y can be found and given by known quantities, in one of the following manners :

Three different methods of resolving equations containing various unknown quantities.

§ 94. First resolve both equations with regard to the same unknown quantities, for example, x , we will have the following two,

$$\text{First method.} \quad (e'') \quad \begin{cases} x = \frac{m}{a} - \frac{b}{a}y \\ x = \frac{n}{l} - \frac{g}{l}y; \end{cases}$$

x , therefore, is equal at once to the second member of the first and of the second equations. Hence, according to the known axiom, the two second members are equal to each other; and consequently form the following equation :

$$\frac{m}{a} - \frac{b}{a}y = \frac{n}{l} - \frac{g}{l}y;$$

from which we derive

$$y = \left(\frac{m}{a} - \frac{n}{l}\right) : \left(\frac{b}{a} - \frac{g}{l}\right) = \frac{ml - an}{bl - ag},$$

the value, namely, of y , given by all known terms. Now, this value being substituted in any of the preceding (e''), suppose in the first, we will obtain the value of the other unknown quantity x ; namely,

$$x = \frac{m}{a} - \frac{b}{a} \times \frac{ml - an}{bl - ag} = \frac{bn - mg}{bl - ag}.$$

Now this and the preceding value of y substituted in the given equations (e) and (e') fulfil the conditions of the same equations, causing the first member to become identical with the second.

Second method.

A second method to find the values of the unknown quantities contained in (e) and (e') consists in resolving only one of the two equations, for instance, (e) with regard to x , which gives, as we have seen,

$$x = \frac{m}{a} - \frac{b}{a}y,$$

and substituting then this value of x in the second equation (e'), in this manner we will have

$$l \times \left(\frac{m}{a} - \frac{b}{a}y\right) + gy = n;$$

from which

$$y = \frac{an - ml}{ag - bl} = \frac{ml - an}{ag - bl}.$$

When we change the sign of both terms a and b of the fraction $\frac{a}{b}$, the sign and value of the whole fraction remains unchanged; because, if both a and b have the same sign, the sign of the quotient is positive, either with a and b positive, or with a and b negative; if a and b have different signs, whatever be the sign of a and b , the quotient is negative. Now, by changing the sign of both terms, if before they were equal, they will

remain so ever after; if they were unequal, such will they be also after the change.

Again, if the difference

$$an - ml = + d,$$

the difference

$$-an + ml,$$

or

$$ml - an = - d;$$

therefore, the fraction

$$\frac{ml - an}{ag - bl}$$

contains the same numerator and denominator of

$$\frac{an - ml}{bl - ag},$$

both with changed signs. Hence,

$$\frac{ml - an}{ag - bl} = \frac{an - ml}{bl - ag};$$

just as we obtained with the preceding process, and making again the substitution as before, we will have the value of x , &c.

Third method.

A third method consists in giving the same coefficient to one of the unknown quantities in both equations, which is easily obtained by multiplying all the terms of (e) by the coefficient, for instance, of x in (e'), and all the terms of (e') by the coefficient of x in e . Thus, we have

$$alx + bly = ml$$

$$alx + gay = na;$$

or else

$$alx + bly - ml = 0$$

$$alx + bay - na = 0.$$

Now, subtracting the second of these equations from the first, the term alx and, consequently, the unknown quantity x will be eliminated; and the only unknown quantity y will remain as follows:

$$bly - gay - ml + na = 0,$$

from which

$$y = \frac{ml - na}{ag - bl},$$

as before.

The same methods of elimination applicable to all cases. § 95. The same modes of elimination are applicable to the case of three unknown quantities variously combined in three different equations; because one of the unknown quantities being eliminated from the first two equations, we obtain another equation with only two unknown quantities; and eliminating, likewise, the same unknown quantity, either from the first and third, or second and third given, we will obtain another equation with the two remaining unknown quantities; and in this manner the case of three unknown quantities contained in three different equations is reduced to the case of two equations and two unknown quantities; from which, the value of the quantities being determined and substituted in any of the given equations, we will find the value of the third unknown quantity. The same thing will be better illustrated by an example. Let us first remark that, with four equations, we can also find the value of four unknown quantities with five, five, and so on. But let the three given equations be as follows:

- Examples.**
- (a) $2x + 3y + 4z = 31.$
 - (b) $3x - y + 2z = 10.$
 - (c) $4x + 3y - 2z = 3.$

What is the value of x , of y , and z ?

In the resolution of equations and problems, practice and even a natural aptitude add considerably to the general rules commonly given, by suggesting expedients and artifices adapted to the various cases. Thus, for example, for the resolution of the given equations it is immaterial to commence the elimination from any of the unknown quantities, but it is more expedient to commence this elimination by the quantity y , and combine the first with the second, and then the second with the third equation. Since, to obtain this elimina-

tion, it is enough to multiply the terms of the second equation (b) by 3, and then add it first to (a) and then to (c). Now (b), multiplied by 3, gives

$$9x - 3y + 6z = 30,$$

and this equation added to (a), gives

$$(d) \quad 11x + 10z = 61,$$

and the same added to (c), gives

$$(e) \quad 13x + 4z = 33.$$

Now from the two, (d) and (e), we can eliminate either of the remaining unknown quantities; but it is more expedient to select z , because, multiplying the terms of (d) by 2, and those of (e) by 5, we will have z affected by the same coefficient in both resulting equations, as follows:

$$22x + 20z = 122$$

$$65x + 20z = 165.$$

Subtract now the first of these two equations from the second, we will have

$$43x = 43,$$

and consequently $x = 1$.

Substitute now this value of x in (d), it will become

$$11 + 10z = 61,$$

from which $z = 5$.

Substitute both x and z , we will have

$$(b) \quad 3 - y + 10 = 10,$$

and consequently, $y = 3$.

§ 96. Although all the unknown quantities
Other examples. to be determined are not always to be found in all the equations employed for this resolution, still the process of the resolution is nearly the same in all cases. Suppose, for instance, that of the three unknown quantities x, y, z , x and y alone are in the first equation, and x and z alone in the second, and y and z in the third. Then eliminating z from

the second and third, we will have an equation containing x and y as the first one, from which both the same x and y will be easily found. For example, let the three equations be

$$x + y = b$$

$$x - z = c$$

$$2y + z = d.$$

Adding the third to the second, we have

$$x + 2y = c + d.$$

Subtracting now the first from this, we obtain

$$y = c + d - b,$$

which, substituted in the first, gives x , and substituted in the third, gives z .

§ 97. Other equations to be resolved :

$$(1.) \quad \begin{cases} 3x + 2y = 118 \\ x + 5y = 191. \end{cases}$$

$$(2.) \quad \begin{cases} 7y = 2x - 3y \\ 19x - 60y = 2485. \end{cases}$$

$$(3.) \quad \begin{cases} x + y = 10 \\ x + z = 19 \\ y + z = 23. \end{cases}$$

$$(4.) \quad \begin{cases} x + y + z = 1\frac{1}{4}7 \\ x + y - z = \frac{7}{4}3 \\ x - y + z = \frac{5}{4}5. \end{cases}$$

$$(5.) \quad \begin{cases} x + y + z = a \\ my = nx \\ pz = qx. \end{cases}$$

$$(6.) \quad \begin{cases} 3x + 5y = 161 \\ 7x + 2z = 209 \\ 2y + z = 89. \end{cases}$$

$$(7.) \quad \begin{cases} y + \frac{1}{2}x = 41 \\ x + \frac{1}{4}z = \frac{4}{2}1 \\ y + \frac{1}{8}z = 34. \end{cases}$$

$$(8.) \quad \begin{cases} 53 - \frac{x}{2} - \frac{y}{2} = y - 109 \\ \frac{x}{4} + \frac{y}{8} = 26 \\ 5y = 4x. \end{cases}$$

$$(9.) \quad \begin{cases} 2x + 5y - 7z + 288 = 0 \\ 5x - y + 3z = 227 \\ 7x + 6y + z = 297. \end{cases}$$

$$(10.) \quad \begin{cases} 18x - 7y - 5z = 11 \\ \frac{2}{5}y - \frac{2}{3}x + z = 108 \\ \frac{7}{2}z + 2y + \frac{3}{4}x = 80. \end{cases}$$

Answers :

$$(1.) \quad x = 16, y = 35.$$

$$(2.) \quad x = 3\frac{5}{4}, y = 7\frac{1}{4}.$$

$$(3.) \quad x = 3, y = 7, z = 16.$$

$$(4.) \quad x = 16, y = 3\frac{1}{4}, z = 1\frac{1}{2}.$$

$$(5.) \quad \begin{cases} x = \frac{amp}{mp + np + mq} \\ y = \frac{anp}{mp + np + mq} \\ z = \frac{amq}{mp + np + mq} \end{cases}$$

$$(6.) \quad x = 17, y = 22, z = 45.$$

$$(7.) \quad x = 18, y = 32, z = 10.$$

$$(8.) \quad x = 64, y = 80, z = 100.$$

$$(9.) \quad x = 13, y = 24, z = 62.$$

$$(10.) \quad x = 12, y = 25, z = 6.$$

§ 98. The third part of a certain number added to the fourth part of a second number, gives a sum equal to two. Again, the sixth part of the first number plus the second number gives a sum equal to four. What are the two numbers ?

Resolution. The two numbers are x and y . The third part

of the first is then $\frac{x}{3}$, and the fourth part of the second $\frac{y}{4}$;

hence, the first equation

$$\frac{x}{3} + \frac{y}{4} = 2,$$

from which we infer

$$x + \frac{3}{4}y = 6.$$

Again, the sixth part of the first number is $\frac{x}{6}$, and the second number is equal to y ; therefore, the second equation is

$$\frac{x}{6} + y = 4,$$

from which

$$x + 6y = 24.$$

Now, subtracting the preceding similar equation from this, we have

$$\left(6 - \frac{3}{4}\right)y = 18,$$

and consequently,

$$y = \frac{24}{7},$$

and substituting this value in the first of the two preceding equations, we have

$$x + \frac{18}{7} = 6,$$

that is,

$$x = \frac{24}{7};$$

hence, both numbers, x and y are, equal to $\frac{24}{7}$.

In fact,

$$\frac{1}{3} \cdot \frac{24}{7} + \frac{1}{4} \cdot \frac{24}{7} = 2$$

$$\frac{1}{6} \cdot \frac{24}{7} + \frac{24}{7} = 4.$$

Find out two numbers whose sum is equal to 70

Problem 2. and difference 16.

Answer:

The equations are

$$x + y = 70$$

$$x - y = 16,$$

and the numbers

$$x = 43, y = 27.$$

Two boxes contain together 300 dollars. If 30 dollars be taken from the first and put in the second, there is the same number of dollars in both of them. What is their contents?

Answer:

The equations are

$$x + y = 300$$

$$x - 30 = y + 30,$$

and

$$x = 180$$

$$y = 120.$$

There is such a fraction that, if we add one to its numerator, its value is equal to $\frac{1}{3}$; and, if we add one to its denominator, its value is equal to $\frac{1}{4}$. What is the fraction?

Answer:

Equations,

$$\frac{x+1}{y} = \frac{1}{3}$$

$$\frac{x}{y+1} = \frac{1}{4},$$

and

$$\frac{x}{y} = \frac{4}{15}.$$

A blacksmith knows that two fifth parts of a mass of iron weigh 96 pounds less than three fourth parts of another mass, and at the same time four ninth parts of the first have precisely the same weight as five eighth parts of the second. What is the weight of each mass?

Answer:

Equations,

$$\frac{3}{4}y - \frac{2}{5}x = 96$$

$$\frac{5}{8}y - \frac{4}{9}x = 0,$$

and

$$x = 720$$

$$y = 512.$$

CHAPTER II.

EQUATIONS OF THE SECOND DEGREE.

Two classes of quadratic equations.

§ 99. EQUATIONS of the second degree, called also *quadratic equations*, are divided into two classes, *incomplete* and *complete*. Those equations in which the square only of the unknown quantities is involved, are of the first description. Complete equations are those in which, besides the square of the unknown quantity, there are also terms with the simple quantity. Hence, the equations

$$x^2 - b = c, \quad 2x^2 - m = n - x^2$$

are incomplete. And

$$x^2 - bx = c, \quad 2x^2 - mx = n - x^2$$

are complete quadratic equations.

ARTICLE I.

Resolution of Incomplete Quadratic Equations.

Resolved as equations of the 1st degree.

§ 100. INCOMPLETE or *pure* quadratic equations are managed exactly as equations of the first degree, as far as the separation of the unknown quantity from all the rest is concerned. So, for example, from the equation

$$2ax^2 - m = q + d,$$

we have

$$x^2 = \frac{q + m + d}{2a},$$

in the same manner as if the square of x would be the simple unknown quantity x .

Square root to be found besides.

But the object of the resolution of quadratic equations and generally of equations of any degree is not that of finding the power of the unknown quantity, but the quantity itself, or the root of that power. Hence, the values of the unknown quantities of these equations are also simply called *roots* of the equations. Therefore, in our case, to complete the resolution, it is necessary to find the square root of $\frac{q + m + d}{2a}$, which is the known value of x .

Double root.

Now the square root of any quantity has a double value, the one positive and the other negative; because $-a$, as well as $+a$, for instance, gives the same positive square a^2 ; and, consequently, the square root of a^2 is $-a$ as well as $+a$, as we noticed already. Designating now by r the square root of $\frac{q + m + d}{2a}$ or x^2 , we will have

$$\sqrt{\frac{q + m + d}{2a}} = \pm r,$$

and, consequently, $x = \pm r$.

The value, therefore, of the unknown quantity in any incomplete quadratic equation is twofold, the one positive and the other negative, both, however, having the same numerical value.

§ 101. Let an equation to be resolved be

$$2x^2 - 4 = 6x^2 - 68.$$

Examples.

We will have first

$$x^2 = \frac{68 - 4}{4} = 16,$$

and then

$$x = \pm \sqrt{16} = \pm 4.$$

Substituting, in fact, either $+4$ or -4 in the given equa-

tion, the first member gives in both cases the same result of the second, namely,

$$28 = 28.$$

Real and rational roots.

Other given equations.

$$(1.) \quad 4x^3 - 7 = 3x^3 + 9.$$

$$(2.) \quad \frac{1}{3}x^3 - 3 + \frac{5}{12}x^3 = \frac{7}{24} - x^3 + \frac{299}{24}.$$

$$(3.) \quad \frac{1}{2}x^3 - 16 - \frac{3}{4}x^3 - 10 = 100 - \frac{9}{4}x^3 + 2.$$

$$(4.) \quad 3x^3 - 8 = 100.$$

$$(5.) \quad \frac{1}{5}x^3 - 5 = 0.$$

Answers :

$$(1.) \quad x = \pm 4.$$

$$(2.) \quad x = \pm 3.$$

$$(3.) \quad x = \pm 8.$$

$$(4.) \quad x = \pm 6.$$

$$(5.) \quad x = \pm 5.$$

Sometimes the number equivalent to the square of the unknown quantity is not a perfect square. Suppose, for example,

$$x^2 = 13.$$

In this case the value of x is represented by placing before the number the radical sign, as follows :

$$x = \pm \sqrt{13}.$$

Examples :

Irrational roots.

$$(1.) \quad 3x^2 - 4 = 1.$$

$$(2.) \quad 7x^2 - 5 = x^2 + 12.$$

$$(3.) \quad \frac{1}{2}x^2 - \frac{1}{4} - 5 = \frac{1}{6}x^2 + 7.$$

$$(4.) \quad 13x^2 + 12 + \frac{1}{4} = 17x^2 - 2 - \frac{3}{4}.$$

$$(5.) \quad 2x^2 - 4 = 9.$$

Answers:

- (1.) $x = \pm \sqrt{\frac{5}{3}}$.
- (2.) $x = \pm \sqrt{\frac{17}{8}}$.
- (3.) $x = \pm \frac{1}{2} \sqrt{147}$.
- (4.) $x = \pm \sqrt{5}$.
- (5.) $x = \pm \sqrt{\frac{13}{2}}$.

Substituting these values in the corresponding equations, we will find the first member of each of them equal to the second.

It may, finally, occur that the resolution of these quadratic equations cannot be obtained by any real root, either rational or irrational, when, namely, the square of the unknown quantity is equivalent to a negative number or value. For example, from the equation

$$2x^2 + 14 = 6,$$

we have

$$x^2 = -4.$$

Now, we have seen (66) that a negative quantity cannot be the square of any real quantity, and generally $\sqrt{-a}$ is expressed also by $\sqrt{a} \cdot \sqrt{-1}$; hence, in our case

$$x = \pm \sqrt{-4} = \pm \sqrt{4} \cdot \sqrt{-1},$$

or

$$x = \pm 2 \sqrt{-1}.$$

And this imaginary value of x substituted in the given equation makes really the first member equal to 6.

Other examples:

- (1.) $\frac{7}{2}x^2 + 14 = -\frac{1}{2}x^2 - 6$.
- (2.) $7x^2 + 18 = 9 - 2x^2$.
- (3.) $12x^2 - \frac{1}{2}x^2 + 20 = \frac{1}{2}x^2 - 2$.
- (4.) $x^2 + 14 = 7$.
- (5.) $2x^2 + \frac{13}{2} = -\frac{1}{4}$.

Answers :

$$(1.) \quad x = \pm 5\sqrt{-1}.$$

$$(2.) \quad x = \pm \sqrt{-1}.$$

$$(3.) \quad x = \pm 2\sqrt{-1}.$$

$$(4.) \quad x = \pm 7\sqrt{-1}.$$

$$(5.) \quad x = \pm \frac{2}{3}\sqrt{-1}.$$

§ 102. Each ring of a golden chain enriched with diamonds, costs as many dollars as there are rings in the chain less four dollars ; and the price of each ring multiplied by the number of rings plus four gives 9984. What is the number of rings ? What the price of each ring ? What the price of the chain ?

From the conditions of the problem it is plain that the answer to the first question affords an easy solution to the second and third.

Let then x be the number of rings. The price of each ring is

$$x-4;$$

this price multiplied by $x+4$ gives a product equal to 9984 ; therefore, $(x-4)(x+4) = 9984,$

and consequently $x^2 - 16 = 9984,$

an equation which resolved gives

$$x = \pm 100.$$

The negative sign is evidently to be excluded ; because, although in the solution of the equation the unknown quantity admits of both signs, yet the nature of the question shows which of the two signs is to be taken. The number of the rings therefore is

$$100,$$

the price of each ring is $100 - 4,$

that is, 96 dollars,

and consequently the price of the chain,

$$9600 \text{ dollars.}$$

Problem 2. A master gives monthly to each of his servants as many pieces of money as there are servants; at the end of the year he has paid 10800 pieces. What is the number of the servants?

Answer :

Equation $12x^2 = 10800,$

from which $x = \pm 30;$

hence, the number of servants is

$$x = \pm 30.$$

Problem 3. What is the number, which multiplied by itself, and divided by 6, gives 864?

Answer :

Equation, $\frac{x^2}{6} = 864;$

hence, $x = \pm 72.$

Problem 4. The product of the eighth part by the seventh part of a number divided by 3 gives $298 + \frac{2}{3}$. What is the number?

Answer :

Equation, $\frac{x^2}{168} = \frac{896}{3}$
 $x = \pm 224.$

Problem 5. Adding a number to 94, and subtracting the same number from 94, multiplying then the sum by the difference, we have 8512. What is the number added and subtracted?

Answer :

Equation, $8836 - x^2 = 8512$
 $x = \pm 18.$

ARTICLE II.

Resolution of Complete Quadratic Equations.

The same cases already considered occur with regard to complete equations.

§ 103. WE have seen (99) what is the difference between complete and incomplete quadratic equations. Let us here remark that in the resolution of these equations the values of the unknown quantity can be real, either rational or irrational, or both imaginary.

Observe now that the first member of the equation

$$(x + a)^2 = m,$$

which is a perfect square, when resolved gives

$$x^2 + 2ax + a^2 = m,$$

the square of the first term plus the double product of the first into the second plus the square of the second term, according to the known law (69).

From this it follows that whenever a trinomial, for instance,

$$x^2 + nx + q,$$

is considered as a perfect square of x plus another quantity, the coefficient n of x must be the double of that other quantity, and q the square of the same quantity, that is, $x + \sqrt{q}$ must be the root of the square.

It follows, besides, that adding to a binomial, for instance,

$$x^2 + qx,$$

the square of half of the coefficient q of x , the binomial will become the perfect square of $x + \frac{1}{2}q$. In fact,

$$\begin{aligned} (x + \tfrac{1}{2}q)^2 &= x^2 + qx + (\tfrac{1}{2}q)^2 \\ &= x^2 + qx + \tfrac{1}{4}q^2. \end{aligned}$$

These things being observed, it is easy to resolve any complete equation of the second degree containing only one unknown quantity.

§ 104. Any equation of the second degree is
General formulas. compendiously expressed, either by

$$x^2 + ax = 0 \quad (a)$$

or

$$x^2 + bx = p \quad (b).$$

That is to say, any equation of the second degree containing only one unknown quantity can be reduced in all cases to one of the two preceding formulas in which the square of the unknown quantity is cleared of the coefficient. The simple unknown quantity, together with its coefficient, is added to the first; and all the other terms, if there are any in which the unknown quantity is not to be found, are brought to the second member.

For example, the equation

$$2x^2 - 3x + \frac{1}{2}x^2 = 4x - \frac{1}{2}x^2,$$

in which no terms are to be found not containing x , is easily transformed into

$$2x^2 + \frac{1}{2}x^2 + \frac{1}{2}x^2 - 3x - 4x = 0;$$

from which

$$(2 + \frac{1}{2} + \frac{1}{2})x^2 - (3 + 4)x = 0,$$

$$3x^2 - 7x = 0,$$

and finally

$$x^2 - \frac{7}{3}x = 0.$$

The same reduction can be done with regard to any other equation of the same kind.

The equation

$$3x^2 - 2x + \frac{1}{4}x^2 + 6 = \frac{3}{4}x^2 - 4x + 9 - \frac{1}{2}x^2,$$

in which, together with the unknown quantities, there are also the known terms 6 and 9, is easily transformed into

$$3x^2 + \frac{1}{4}x^2 - \frac{3}{4}x^2 + \frac{1}{2}x^2 - 2x + 4x = 9 - 6,$$

from which

$$3x^2 + 2x = 3,$$

and

$$x^2 + \frac{2}{3}x = 1.$$

The same may be said of any other like equation.

The two formulas therefore mentioned above represent all the equations of the second degree involving only one unknown quantity. Hence, giving the rule to resolve the two formulas, with the same rule we are able to resolve all the complete equations of the second degree.

Solution of the first formula.

§ 105. The resolution of the second general formulas is applicable also to the first; but with regard to the first, observe that (a) can be reduced to an equation of the first degree, since, dividing both terms by x , we obtain

$$x + a = 0,$$

and from this

$$x = -a.$$

But the equation of the first degree gives only one value of the unknown quantity, while the equation of the second degree admits two.

From (a) , according to the 2d axiom, (85,) we may infer the following equation

$$x^2 + ax + g = g,$$

g having any value whatsoever. Now, if we make $g = (\frac{1}{2}a)^2$ equal to the square of one-half the coefficient of x , the first member becomes the perfect square of $x + \frac{1}{2}a$, and we will have

$$x^2 + ax + \frac{1}{4}a^2 = \frac{1}{4}a^2.$$

And when this addition is made we *complete the square*. But with this addition, as we have remarked, the first member of the equation is $(x + \frac{1}{2}a)^2$; therefore,

$$(x + \frac{1}{2}a)^2 = \frac{1}{4}a^2,$$

and consequently

$$\sqrt{(x + \frac{1}{2}a)^2} = \pm \sqrt{\frac{1}{4}a^2},$$

that is,

$$x + \frac{1}{2}a = \pm \frac{1}{2}a,$$

and consequently

$$x = -\frac{1}{2}a + \frac{1}{2}a = 0$$

$$x = -\frac{1}{2}a - \frac{1}{2}a = -a.$$

That is to say, when the equation *reduced* takes the form of (a), one of the values of the unknown quantity is equal to zero, and the other equal to the coefficient of x taken with the opposite sign.

Rule. To resolve any complete equation of the second degree not involving terms without the unknown quantity, *reduce the equation to the simplest form, and make the second member equal to zero, divide by the coefficient of the square of the unknown quantity, and the coefficient of the second term taken with the opposite sign will be one of the roots of the unknown quantity, the other root being equal to zero.*

Examples. Let a given equation be

$$3x^2 - \frac{1}{2}x + \frac{1}{4}x^2 = 2x^2 + \frac{1}{2}x;$$

the same reduced gives

$$\frac{5}{4}x^2 - x = 0,$$

and clearing the first term of the coefficient

$$x^2 - \frac{4}{5}x = 0.$$

The roots, therefore, or values of the unknown quantity are

$$x = 0, x = +\frac{4}{5}.$$

Substituting, in fact, zero instead of x , the first and second members of the given equation become both equal to zero, and substituting for x , $(+\frac{4}{5})$, we will have

$$3.(+\frac{4}{5})^2 - \frac{1}{2}(+\frac{4}{5}) + \frac{1}{4}(+\frac{4}{5})^2 = 2.(+\frac{4}{5})^2 + \frac{1}{2}(+\frac{4}{5});$$

and reducing separately each member, we have

$$\frac{168}{100} = \frac{168}{100}.$$

Other given equations:

$$(1.) \quad 3x^2 - \frac{1}{2}x^2 = 2x + \frac{1}{3}x^2.$$

$$(2.) \quad \frac{1}{4}x^2 - \frac{1}{4}x = 2x^2 - \frac{1}{2}x.$$

$$(3.) \quad \frac{9}{4}x^2 - 2x + \frac{1}{4}x^2 = \frac{1}{2}x^2 - \frac{1}{4}x^2 - 6x.$$

$$(4.) \quad \frac{3}{4}x + \frac{4}{5}x^2 = \frac{6}{5}x + \frac{9}{4}x^2.$$

$$(5.) \quad x^2 - 2x = -17x - 4x^2.$$

Answers :

$$(1.) \quad x = 0, \quad x = +\frac{1}{3}.$$

$$(2.) \quad x = 0, \quad x = +\frac{1}{4}.$$

$$(3.) \quad x = 0, \quad x = -8.$$

$$(4.) \quad x = 0, \quad x = -\frac{3}{2}.$$

$$(5.) \quad x = 0, \quad x = -3.$$

Solution of the
second formula.

§ 106. When the complete quadratic equation involves terms without the unknown quantity, it may always be reduced, as we have seen in the form of (b), (104.) The process, therefore, to be pursued to resolve (b) is the same for all other equations of which (b) is the general formula. Now, to resolve (b), complete first the square of the first member, as for (a), adding, namely, to both members of (b) $\frac{1}{4}b^2$, which is the square of one-half of the coefficient of x ; in this manner we have

$$x^2 + bx + \frac{1}{4}b^2 = p + \frac{1}{4}b^2,$$

that is

$$(x + \frac{1}{2}b)^2 = p + \frac{1}{4}b^2,$$

and consequently,

$$\sqrt{(x + \frac{1}{2}b)^2} = \pm \sqrt{p + \frac{1}{4}b^2},$$

or else

$$x + \frac{1}{2}b = \pm \sqrt{p + \frac{1}{4}b^2},$$

from which

$$x = -\frac{1}{2}b + \sqrt{p + \frac{1}{4}b^2}$$

$$x = -\frac{1}{2}b - \sqrt{p + \frac{1}{4}b^2}.$$

Rule.

And, consequently, when the given equation involves terms excluding the unknown quantities, the rule to be followed in resolving, is this :

Bring all the terms containing the unknown quantity to the first member, and all the others to the second, reduce the terms of the first member to two, the first with the square, the second with the simple unknown quantity, clear the first of these terms of its coefficient, complete the square. Take then the square root of both members, from which arises an equation of the first degree with double sign, to be resolved as usual.

Example. Let, for example, the equation to be resolved be

$$3x^2 + 2x + 8 = 16 + 4x + 2x^2.$$

Bringing all the terms with the unknown x to the first member, and the others to the second, we will have

$$3x^2 + 2x - 4x - 2x^2 = 16 - 8,$$

which reduced gives

$$x^2 - 2x = 8,$$

with the coefficient of the first term equal to unity. The coefficient of the second term is -2 , which divided by 2, gives -1 ; and $+1$ being the square of -1 , we will have the square completed, as follows :

$$x^2 - 2x + 1 = 8 + 1 = 9,$$

that is,

$$(x - 1)^2 = 9,$$

and consequently,

$$x - 1 = \pm\sqrt{9} = \pm 3,$$

from which

$$x = +1 + 3 = 4$$

$$x = +1 - 3 = -2.$$

Substituting now, in the given equation, $+4$ instead of x , we find

$$64 = 64;$$

and, substituting -2 instead of x ,

$$16 = 16;$$

both values, therefore, $+4$ and -2 , satisfy the equation.

Other examples. § 107. Given equations :

$$(1.) \quad x^2 + 6x = 27.$$

$$(2.) \quad x^2 - 7x + \frac{1}{4} = 0.$$

$$(3.) \quad x^2 - \frac{3}{4}x = 18.$$

$$(4.) \quad 3x^2 - 2x = 65.$$

$$(5.) \quad 622x = 15x^2 + 6384.$$

$$(6.) \quad \frac{4}{5}x - \frac{3}{16} = x^2.$$

$$(7.) \quad \frac{2}{3}x^2 - \frac{2}{3}x + 195 = 0.$$

$$(8.) \frac{1}{5}x^3 + \frac{1807}{5}x + 4728 = 0.$$

$$(9.) x^3 - 8x = 14.$$

$$(10.) 118x - \frac{5}{3}x^3 = 20.$$

Answers :

$$(1.) x = 3, x = -9.$$

$$(2.) x = \frac{1}{2}^3, x = \frac{1}{2}.$$

$$(3.) x = 8, x = -\frac{3}{4}.$$

$$(4.) x = 5, x = -\frac{1}{3}^3.$$

$$(5.) x = \frac{1}{5}^4, x = \frac{5}{3}^6.$$

$$(6.) x = \frac{1}{2}^7, x = \frac{1}{4}^5.$$

$$(7.) x = \frac{1}{7}^5, x = \frac{1}{4}^3.$$

$$(8.) x = -(25 + \frac{1}{3}^9), x = -52.$$

$$(9.) x = 9,4772, x = -1,4772.$$

$$(10.) x = 47,0298, x = 0,1701.$$

Irrational roots. These two last examples deserve to be noticed, because they afford examples of irrational roots by which the equations are resolved. The equation (9) resolved as usual, gives

$$x = 4 \pm \sqrt{30}.$$

The square root of 30 cannot be found except by approximation, adding a certain number of decimal figures to the number 30, which gives

$$\sqrt{30,0000} = 5,4772;$$

hence,

$$x = 9,4772$$

$$x = -1,4772.$$

A similar remark can be made with regard to the last example.

Imaginary roots. It remains, finally, to see how sometimes the roots are imaginary.

Take the equation,

$$8x^3 - 7x + 34 = 0,$$

which, resolved according to the rule, gives

$$\left(x - \frac{7}{16}\right)^2 = -\frac{1039}{(16)^2};$$

hence,

$$x = \frac{7}{16} \pm \sqrt{-\frac{1039}{16^2}} = \frac{7}{16} \pm \frac{\sqrt{1039} \cdot \sqrt{-1}}{16}$$

Problem 1. § 108. The square of the seventh part of the miles run over by a steamboat in two hours being added to the miles, and the number 100 being taken from the sum, we have a remainder equal to 10. What is the number of miles for each hour?

Solution. Call x the seventh part of the miles run over by the steamboat.

The number of the miles will be

$$7x;$$

this number added to x^2 , gives the sum,

$$x^2 + 7x,$$

from which taking 110, we ought to have 10. The equation therefore included in the problem is

$$x^2 + 7x - 110 = 10,$$

which resolved, gives first

$$x + \frac{7}{2} = \pm \sqrt{120 + \frac{49}{4}},$$

and then
$$x = -\frac{7}{2} \pm \sqrt{\frac{529}{4}};$$

hence,
$$x = -\frac{7}{2} + \frac{23}{2} = +8$$

$$x = -\frac{7}{2} - \frac{23}{2} = -15.$$

The negative value is to be excluded. Therefore,

$$7x = 56.$$

And the number of miles run over each hour being one-half of $7x$, is

Problem 2. A general of an army takes out from four equal square battalions, sixteen ranks, and 120 men besides; the number of soldiers remaining is 3000. How many men are in each rank? How many in each battalion?

The answer to the second question is immediately deduced from that given to the first, because the number of men of each rank being x , the number of men of each battalion is x^2 .

Answer:

The equation is $4x^2 - 16x - 120 = 3000$,

and $x - 2 = \pm \sqrt{784}$

$$x = 2 \pm 28$$

$$x = 30, \quad x = -26.$$

The second value is to be excluded. The number of men in each rank is then

$$x = 30.$$

That of the men in each battalion,

$$x^2 = 900.$$

Problem 3. A man lost in a game twenty-one times the square root of the number of his dollars, and 90 dollars besides; after the game he finds that 180 dollars still remain in his pocket. How many dollars had he before commencing the game?

Answer:

Equation, $x^2 - 21x - 90 = 180$

$$x = \frac{21}{2} + \sqrt{\frac{1521}{4}}$$

$$x = \frac{21 + 39}{2} = +30$$

$$x = \frac{21 - 39}{2} = -9.$$

The second value is to be excluded; therefore, the number of dollars before the game is

$$x^2 = 900.$$

Problem 4. If two pounds be subtracted from one-third of the square number of pounds of a mass of silver, we will have a negative difference equal to one-ninth of the weight of the same mass, and $\frac{1}{3}$ besides. What is the weight of the mass of silver.

Answer:

Equation, $\frac{1}{3}x^2 - 2 = -\frac{1}{9}x - \frac{1}{3},$

from which $x = -\frac{1}{3} + \frac{1}{3}^2 = 2$

$x = -\frac{1}{3} - \frac{1}{3}^2 = -\frac{7}{3}.$

The second number being excluded, we have

$x = 2$ pounds.

Problem 5. The square of a number, being taken twelve times, is equal to four times the number plus 8. What is the number?

Answer:

Equation, $12x^2 = 4x + 8,$

from which $x = 1$

$x = -\frac{1}{3},$

both of them satisfying the equation.

CHAPTER III.

PROPORTIONS AND PROGRESSIONS.

§ 109. So frequent and so important is the use of proportions in almost all mathematical branches either of pure mathematics or of mathematics applied to physics, that we cannot omit to give as early as possible to the young pupil some idea of them. And no more than this is that which we are about to do in this last chapter.

Arithmetical and geometrical ratios and proportions. Technical terms.

The difference $a - b$,
between any two quantities a and b , is called sometimes *arithmetical ratio*; and the first term a is called *antecedent*, the second, b , *consequent*; when two such ratios are equal to one another, for example, $a - b$, and $a' - b'$, the equation

$$a - b = a' - b'$$

is called also *arithmetical proportion*. The first and last terms of which are called the *extreme* terms; and the second b , and third a' , the *means*.

Likewise, the quotient $\frac{m}{n}$

is called sometimes *geometrical ratio*; the numerator m is called *antecedent*, and the denominator n , *consequent*. And when two such ratios are equal to one another, for instance,

$\frac{m}{n}$ and $\frac{m'}{n'}$, the equation

$$\frac{m}{n} = \frac{m'}{n'}$$

is called also *geometrical proportion*, which more commonly is written as follows:

$$m : n :: m' : n'.$$

And again, the first and last terms are the *extremes*, the other two the *means*. Geometrical proportion, and also arithmetical, is read as follows:

m is to n , as m' is to n' ;

that is, the two dots : or the sign — are read *is to*, and the sign :: or = is read *as*.

ARTICLE I.

Arithmetical Proportions and Progressions.

How an arithmeti-
cal proportion may
be modified.

§ 110. THE arithmetical proportion

$$a - b = a' - b'$$

can be modified in different manners without changing the value or equality of the two ratios. For example, we may add or subtract from both terms of one ratio (and even of both) the same quantity without changing the ratios and their equality, or we may add either to the *antecedent* or to the *consequent* of each ratio the same quantity, the first ratio still remaining equal to the second.

So, for example, if we add 3 to the antecedent of both ratios of the following proportion,

$$6 - 2 = 7 - 3,$$

we will have $6 + 3 = 9$, $7 + 3 = 10$,

and still $9 - 2 = 10 - 3$.

Resuming now the algebraical proportion $a - b = a' - b'$, let us add to both a and a' ($b + b'$), we have

$$(a + b + b') - b = (a' + b + b') - b'.$$

Now $a + b + b' - b = a + b'$,

and $a' + b + b' - b' = a' + b$.

Therefore, from $a - b = a' - b'$,

we infer, $a + b' = b + a'$.

That is, in any arithmetical proportion, the sum of the extremes is equal to that of the means.

Continual pro-
portions.

§ 111. Suppose now that the consequent of the first ratio becomes equal to the antecedent

of the second, the proportion then will be

$$a - b = b - b',$$

and this proportion is called *continual*.

The sum of the ~~mean~~ mean terms is in this case $2b$; hence, we have, according to the preceding inference,

$$2 = a + b',$$

that is

$$b = \frac{a + b'}{2}$$

and this expression equivalent to b is called *mean arithmetical proportional* between a and b' .

§ 112. In the same manner in which $b - b'$ is the second ratio of the continual proportion $a - b = b - b'$, it may be the first ratio of another continual proportion

$$b - b' = b' - b'';$$

and $b' - b''$ likewise can be the first ratio of the proportion

$$b' - b'' = b'' - b''',$$

and so on. This being the case, all the preceding ratios are equal, and we may write

$$a - b = b - b' = b' - b'' = b'' - b''', \text{ \&c.}$$

And the terms $a, b, b', b'', b''', \text{ \&c.}$, of this continual proportion, are said to form an *arithmetical progression*.

Let us now suppose the terms $a, b, b' \dots$ of the progression to increase, and call $-d$ the difference or ratio $a - b$, $-d$ will likewise represent all the following ratios equal to $a - b$; hence, we will have

$$a - b = -d$$

$$b - b' = -d$$

$$b' - b'' = -d, \text{ \&c.,}$$

and consequently, also,

$$b = a + d$$

$$b' = b + d = (a + d) + d = a + 2d$$

$$b'' = b' + d = (a + 2d) + d = a + 3d, \text{ \&c.;}$$

therefore, the terms of any arithmetical progression are represented by

$$a, a + d, a + 2d, a + 3d, a + 4d, \dots (r).$$

Examples. Let us make a and d equal to unity; the terms of the progression will become

$$1, 2, 3, 4, 5, \dots;$$

that is, the series of natural numbers constitutes an arithmetical progression.

Make $a = 1$ and $d = 3$, we will have the numerical and arithmetical progression

$$1, 4, 7, 10, 13, 16, \dots.$$

The terms (r) exhibit the general formula of any arithmetical progression.

ARTICLE II.

Geometrical Proportions and Progressions.

A proportion changed into the form of an equation and vice versa.

§ 113. THE geometrical proportion

$$m : n :: m' : n' \dots (e),$$

as we have remarked, is nothing else than the

equation
$$\frac{m}{n} = \frac{m'}{n'} \dots (e');$$

and from any equation of this form we can always infer a geometrical proportion.

1st Corollary. Let us now observe that from (e') we have

$$\frac{m}{m'} = \frac{n}{n'};$$

hence, also, $m : m' :: n : n'.$

Comparing now this proportion with (e), we see that

The antecedent of the first ratio of any given geometrical proportion is to the antecedent of the second ratio as the consequent of the first is to the consequent of the second.

2d Corollary. From the same (e'), we have also

$$\frac{n}{m} = \frac{n'}{m'},$$

and consequently, $n : m :: n' : m'$;

and comparing this with (e), we conclude that

The consequent of the first ratio of any geometrical proportion is to its antecedent as the consequent of the second is to its own antecedent.

3d Corollary. From (e') again we have

$$\frac{m}{n} + 1 = \frac{m'}{n'} + 1,$$

and
$$\frac{m}{n} - 1 = \frac{m'}{n'} - 1;$$

or else
$$1 + \frac{m}{n} = 1 + \frac{m'}{n'},$$

and
$$1 - \frac{m}{n} = 1 - \frac{m'}{n'}.$$

Reducing the terms of each member of these four equations to the same denominator, we will have

$$\frac{m + n}{n} = \frac{m' + n'}{n'}$$

$$\frac{m - n}{n} = \frac{m' - n'}{n'}$$

$$\frac{n + m}{n} = \frac{n' + m'}{n'}$$

$$\frac{n - m}{n} = \frac{n' - m'}{n'};$$

and consequently,

$$(m + n) : n :: (m' + n') : n'$$

$$(m - n) : n :: (m' - n') : n'$$

$$(n + m) : n :: (n' + m') : n'$$

$$(n - m) : n :: (n' - m') : n'.$$

From which, compared with (e), we infer that

The sum or difference of the terms of the first ratio is to the second term or consequent of the same ratio as the sum or difference of the terms of the second ratio is to the consequent of the second ratio.

Again, from the last proportions and according to the first corollary we have

$$(m + n) : (m' + n') :: n : n'$$

$$(m - n) : (m' - n') :: n : n'$$

$$(n + m) : (n' + m') :: n : n'$$

$$(n - m) : (n' - m') :: n : n'.$$

Therefore, *the sum or difference of the terms of the first ratio is to the sum or difference of the terms of the second ratio as the consequent of the first ratio is to the consequent of the second.*

The fifth and sixth corollary are the same as the third and fourth with regard to the sum and difference of the terms of the ratios. But the consequents n and n' are changed into the antecedents m and m' . This change may be obtained in different ways. But changing (e') according to the second corollary, taking, namely,

$$\frac{n}{m} = \frac{n'}{m'};$$

from this we will have the equations and then the proportions analogous to those obtained for the third corollary, exactly in the same manner. We will, namely, have

$$n + m : m :: n' + m' : m'$$

$$n - m : m :: n' - m' : m'$$

$$m + n : m :: m' + n' : m'$$

$$m - n : m :: m' - n' : m'.$$

Consequently, *the sum or difference of the terms of the first ratio is to the antecedent of that ratio as the sum or difference of the terms of the second ratio is to the antecedent of the same ratio.*

And since from the last proportion we have

$$\begin{aligned} n + m : n' + m' &:: m : m' \\ n - m : n' - m' &:: m : m' \\ m + n : m' + n' &:: m : m' \\ m - n : m' - n' &:: m : m'. \end{aligned}$$

so, the sum or difference of the terms of the first ratio is to the sum or difference of the terms of the second ratio as the antecedent of the first is to the antecedent of the second.

7th Corollary. From the last two proportions we have

$$\begin{aligned} \frac{m+n}{m'+n'} &= \frac{m}{m'} \\ \frac{m-n}{m'-n'} &= \frac{m}{m'}; \end{aligned}$$

therefore,
$$\frac{m+n}{m'+n'} = \frac{m-n}{m'-n'},$$

and also,
$$\frac{m+n}{m-n} = \frac{m'+n'}{m'-n'};$$

hence,

$$\begin{aligned} m + n : m' + n' &:: m - n : m' - n' \\ m + n : m - n &:: m' + n' : m' - n'; \end{aligned}$$

that is to say, making the usual comparison with (e),

The sum of the terms of the first ratio is to the sum of the terms of the second as the difference of the terms of the first ratio is to the difference of the terms of the second.

And again :

The sum of the terms of the first ratio is to their difference as the sum of the terms of the second ratio is to their difference.

Product of the mean and extreme terms. § 114. In the arithmetical proportion, the sum of the extremes is equal to that of the means. And in any geometrical proportion,

$$m : n :: m' : n';$$

the product $m \cdot n'$ of the extreme terms is equal to $n \cdot m'$ the product of the means.

Because, from the equation

$$\frac{m}{n} = \frac{m'}{n'},$$

equivalent to the proportion, we have

$$mn' = nm'.$$

§ 115. Let us make the consequent of the first ratio equal to the antecedent of the second. The proportion then becomes *continual*, namely,

$$m : n :: n : n',$$

from which

$$n \cdot n = m \cdot n',$$

or

$$n^2 = mn'$$

and

$$n = \sqrt{mn'};$$

and this radical expression equal to n is termed *mean geometrical proportional* between m and n' .

§ 116. Let us again suppose that the second ratio of a continual proportion is also the first ratio of another geometrical proportion, and the second ratio of this the first of a third proportion, and so on, so that we may have

$$m : n :: n' : n'' :: n'' : n''' :: n''' : n''', \&c.,$$

or, which is the same,

$$\frac{m}{n} = \frac{n}{n'} = \frac{n'}{n''}, \&c.$$

In this case the terms $m, n, n', \&c.$, constitute a *geometrical progression*. Such terms, namely, that the first divided by the second, and the second divided by the third, and this by the fourth, &c., give always the same quotient, as the terms of an arithmetical progression subtracted in succession from one another give constantly the same difference. Let us call q this constant quotient; we will have

$$\frac{m}{n} = q$$

$$\frac{n}{n'} = q$$

$$\frac{n'}{n''} = q, \text{ \&c.};$$

and, consequently,

$$m = qn$$

$$n = qn'$$

$$n' = qn'', \text{ \&c.},$$

and also,

$$n = \frac{m}{q}$$

$$n' = \frac{n}{q} = \frac{m}{q \cdot q} = \frac{m}{q^2}$$

$$n'' = \frac{n'}{q} = \frac{m}{q \cdot q^2} = \frac{m}{q^3};$$

and in like manner we would find

$$n''' = \frac{m}{q^4}, \quad n'''' = \frac{m}{q^5}, \quad \text{\&c.}$$

Therefore, since $m, n, n', n'' \dots$ constitute the terms of a geometrical progression, the same will be done also, by

$$m, \frac{m}{q}, \frac{m}{q^2}, \frac{m}{q^3}, \text{ \&c.},$$

and making $\frac{1}{q} = z$. Since then $\frac{1}{q^2} = z^2, \frac{1}{q^3} = z^3 \dots$ and $\frac{m}{q} = m \cdot \frac{1}{q}, \frac{m}{q^2} = m \cdot \frac{1}{q^2}, \frac{m}{q^3} = m \cdot \frac{1}{q^3} \dots$

the terms of the same progression will be given by

$$m, mz, mz^2, mz^3, mz^4 \dots$$

Making now $m = 1, z = 2$, we will have the geometrical progression

$$1, 2, 4, 8, 16, 32 \dots$$

and making $m = 2, z = 3$, we will have

$$2, 6, 18, 54, 162 \dots$$

PROBLEMS.

1st. One half of a given number divided by 3 gives 3 for quotient. What is the number. Ans. $x = 18$.

2d. What time is it? asks a man of another, who answers, Three-quarters of the hour just struck are equal to two-thirds of the next hour. What is the number denoting the hour just struck. Ans. $x = 8$.

3d. What is the number to which adding three times its own half minus 3, gives 3 for the sum? Ans. $x = 1\frac{2}{5}$.

4th. If we take 3 from a number, we have a remainder that is equal to one-half the same we shall obtain by adding 6 to the same number. What is the number?

Ans. $x = 12$.

5th. There is a certain number of apples in a basket. A boy takes one-half of them, and half an apple besides. A second boy takes one-half of those that remain, and half an apple besides. A third boy takes one half of the rest, and half an apple. After this two apples still remain in the basket. How many apples were there in the basket?

Ans. $x = 32$.

6th. Take the sum of the ciphers of one-half of my age, you will have 10; subtract the first from the second, you will have 8. What is my age.

Ans. $x = 38$.

7th. A man travels for three days; the first day twice as much as the second; and if the number of miles of the last day be taken from the miles of the second, we have 5 for remainder; if from those of the first, the remainder is 20. How many miles did he travel the first day? Ans. 30.

8th. Two numbers added together give 38, and their difference is 18. What are these numbers?

Ans. $x = 28$, $y = 10$.

9th. Three numbers added together give 9 for sum; the sum of the first added to the last multiplied by two gives four times the second; and the last minus the sum of the second,

and one-half of the first, gives a remainder equal to zero. What are the three numbers? Ans. $x=2$, $y=3$, $z=4$.

10th. A number is composed of three such figures that their product gives 54; the sum of the first and last, divided by the middle one, gives 6 for quotient; and if we subtract from the unknown number the other 594, the remainder is composed of the same three figures, but with inverted order. What is the number. Ans. 923.

11th. A young man having been asked his age, and the age of his father and grandfather, answered, My age together with that of my father makes 56 years, and the age of my father added to that of my grandfather makes 100 years; my age with that of my grandfather makes 80 years. What is the age of each? Ans. $x=18$, $y=38$, $z=62$.

12th. A basin is filled by three fountains; the first alone would fill it in one hour and one-third of an hour, the second alone would fill the basin in three hours and one-third of an hour, and the last in five hours. What interval of time will be required to have the basin filled by the three fountains together? Ans. $\frac{4}{5}$ of an hour.

13th. Three times the square of a number added to the simple number taken four times, gives a sum equal to four times the square of the same number. What is the number? Ans. $x=4$.

14th. The height in feet of a tower added to the square number of feet of the periphery of its base (which number is one-fifth of the height of the tower) gives the sum equal to 500. What is the height of the tower? Ans. 100 feet.

15th. Subtract from the square of a number plus its own root the number 2580, and you will have 3000. What is the number. Ans. $x=180$.

16th. Half the sum of the terms of an arithmetical ratio is equal to 7; one-fourth of the same sum minus $\frac{1}{2}$ is equal to half the ratio. What is the ratio? Ans. $10-4$.

17th. $6-2$ is the first ratio of an arithmetical proportion, and 8 is the last extreme. What is the antecedent of the second ratio?
Ans. $x=12$.

18th. The mean proportional term of an arithmetical proportion having 7 for ratio is 21. What is the proportion?
Ans. $14 : 21 :: 21 : 28$.

19th. The second ratio of a geometrical proportion is $18 : 6$, and the first extreme is 17. What is the consequent of the first ratio?
Ans. $x=9$.

20th. The product of the extreme terms of continual geometrical proportion is 64, the first of them is equal to 2. Find the proportion.
Ans. $2 : 8 :: 8 : 32$.

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